Numerical capture of shock solutions of nonconservative hyperbolic systems via kinetic functions

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Abstract

This paper reviews recent contributions to the numerical approximation of solutions of nonconservative hyperbolic systems with singular viscous perturbations. Various PDE models for complex compressible materials enter the proposed framework. Due to lack of a conservative form in the limit systems, associated weak solutions are known to heavily depend on the underlying viscous regularization. This small scales sensitiveness drives the classical approximate Riemann solvers to grossly fail in the capture of shock solutions. Here, small scales sensitiveness is encoded thanks to the notion of kinetic functions so as to consider a set of generalized jum conditions. To enforce for validity these jump conditions at the discrete level, we describe a systematic and effective correction procedure. Numerical experiments assess the relevance of the proposed method.

1 Introduction

We survey some of the recent numerical methods for approximating the solutions of nonlinear hyperbolic systems with viscous perturbations, in the form :

$$\mathcal{A}_0(\mathbf{v}_\epsilon)\partial_t \mathbf{v}_\epsilon + \mathcal{A}_1(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon = \epsilon \partial_x(\mathcal{D}(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon), \quad x \in I\!R,\ t > 0. \tag{1}$$

Here, the central issue stems from that neither $\mathcal{A}_0$ nor $\mathcal{A}_1$ coincide with Jacobian matrices so that the nonlinear PDE model (1) does not take the standard form of systems in conservation form. Proeminent models from the Physics of complex compressible materials actually enter the present framework. The nonconservative terms in (1) are generically the by-product of simplifying modelling assumptions. In most instances, these assumptions intend to bypass the need

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for dealing with intricate mechanisms taking place at a too fine scale. Averaged multiphase flows [26], [27] or averaged turbulent flows [1], [6] provide typical major examples. Also do plasmas models when scales smaller than the Debye lenght are neglected [17], [14]. At last, a recent multifluid model [15] falls into the present setting. After [14], [1], [6], a surprising property met by most (if not all) of these models stays in the existence of an admissible change of variable such that (1) recasts in the form :

$$\partial_t \mathbf{u}_\epsilon + \partial_x \mathcal{F}(\mathbf{u}_\epsilon) = \epsilon \mathcal{R}(\mathbf{u}_\epsilon, \partial_x \mathbf{u}_\epsilon, \partial_{xx} \mathbf{u}_\epsilon), \quad x \in I\!R,\ t > 0, \tag{2}$$

where in contrast with (3), the regularization term now stands in (genuine) nonconservative form while the left hand side is conservative. Let us stress from now on that this property will play a central role hereafter in view of the difficulties we now enter.

In realistic applications, all the reported models have to be tackled in the regime of a large Reynolds number : *i.e.* the rescaling parameter $\epsilon > 0$ in (1) is small, with typical order of magnitude 10^{-6}. In view of the genuine nonlinearities in the underlying hyperbolic operator in (1), solutions under consideration involve in general propagating viscous shock layers which differ from their end states $\mathbf{u}_-$ and $\mathbf{u}_+$ only in a $\mathcal{O}(\epsilon)$-interval of stiff transition. Obviously, mesh refinements of practical interest cannot afford for a proper resolution of such small scales. Hence and away from solid boundaries, we are urged to consider the singular limit $\epsilon \to 0^+$ in (1). Due to the lack of conservative form, discontinuous limit solutions cannot be understood in the classical sense of distributions. However for these limit solutions, the nonconservative terms in the limit system of (1) exhibit products of discontinuous functions with measures and these must be given some suitable meaning. This difficulty has received several significant contributions over the past decade after the works by LeFloch [21], Dal Maso, LeFloch and Murat [18] to tackle the singular limit in (1) and by Berthon, Coquel and LeFloch [5] to handle the limit system in the distinct but equivalent form (2). We also refer to Colombeau [11], Colombeau, Leroux [12] for a theory in a distinct functionnal framework.

These suitable theories inevitably come with the property that shock solutions in the limit system are inherently regularization dependent : two distinct viscous mechanisms in (1) or equivalently in (2) generically give birth to two different families of shock solutions in the limit system. The small-scales sensitiveness in the discontinuous solutions is in complete opposition with the undependence property met in the usual conservative setting. After [21], [18], we hereafter shade light on the roots of such a sensitivity. In [21], [18], sensitiveness is encoded in terms of a fixed family of paths connecting possible end states in the viscous shock profiles in (1) while in [5], it is encoded in the so-called kinetic functions. Roughly speaking, these kinetic-functions can be regarded as the mass of bounded Borel measures concentrated on the shock solutions to (2) so as to give rise to generalized Rankine-Hugoniot jump conditions. More precisely, the mass of these bounded measures actually coincide with entropy dissipation rates coming with the viscous shock profiles of(2). Both approaches complement each other and are introduced in this review since they are involved in the numerical procedures to be discussed.

These theories are exemplified, in this paper, on an important class of models encompassing several turbulence and multifluid as well modellings. We refer the reader to [1] for the so-called (k, ε) model and variants of it, to [6] for the multi-scales turbulence approach and to [15] for multifluid descriptions. All these models naturally take the form (2) for extended Navier-Stokes equations when considering several independent specific entropies for governing independent pressure laws. The limit system is seen to yield an natural extension of the classical Euler equations involving bounded Borel measures concentrated along the discontinuity curves of the solutions and vanishing everywhere else.

The inherent small scale sensitiveness of weak solutions for the limit system in (1) or (2) makes their numerical approximation a particularly challenging issue. The core of the difficulty indeed stems from the property of shock solutions to be regularization dependent : the artificial dissipation terms induced by numerical methods tend to corrupt the discrete shocks. Large failures in the celebrated Godunov method in the proper capture of shock solutions to (58) are well exemplified in Chalons [6], approximate Riemann solvers grossly fail as well as illustrated in Berthon [1] and Chalons [6]. We refer to Hou, LeFloch [20] for an analysis in the scalar setting. By contrast, the Glimm method stays free from artificial diffusion and has been shown to converge to the correct solutions [23]. Our main purpose in this paper is to illustrate, after [3], [9] and the related works we quote hereafter, how to enforce roughly speaking the artificial diffusion in classical numerical methods to mimic the exact dissipation mechanism. More precisely, the procedure we describe intends to keep all the independent discrete rate of entropy dissipations in the exact balance prescribed by the kinetic functions coming with (2). We refer the reader to Tadmor [28] for a precise link between numerical viscosity and discrete entropy rates. As a consequence of preserved balances, a far much better agreement is achieved between exact and discrete solutions : errors are virtually negligeable for shocks of moderate amplitude in the models investigated in [1], [6].

The format of the present paper is as follows. The second section highlights the roots in the small scales sensitiveness of shock solutions of (1) and (2) as well to then introduce both theories of family of paths and kinetic functions in the class of piecewise Lipshitz continuous functions. The third section describes the main properties of the extended Navier-Stokes equations and their limit model, the extended Euler equations, with a special emphasis put on the description of full sets of extended generalized Rankine-Hugoniot jump relations respectively derived from the two theories. The last section then explains the origin in the failure of classical Riemann solvers so as to naturally introduce a systematic and effective correction procedure.

2 Shock solutions for nonconservative hyperbolic systems

Given a smooth matrix-valued function $\mathcal{D} \geq 0$, this section introduces some of the mathematical tools, developped over the past decade, to handle first order systems with singular viscous perturbation built from $\mathcal{D}$:

$$\mathcal{A}_0(\mathbf{v}_\epsilon)\partial_t \mathbf{v}_\epsilon + \mathcal{A}_1(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon = \epsilon \partial_x(\mathcal{D}(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon). \quad x \in I\!R,\ t > 0, \tag{3}$$

By singular, it is classically meant that (3) is addressed in the regime of a vanishing rescaling parameter $\epsilon \to 0^+$. Here, the unknown $\mathbf{v}_\epsilon$ belongs to some open convex subset $\Omega_\mathbf{v} \subset I\!R^n$. The matrix-valued functions $\mathcal{A}_i : \Omega_\mathbf{v} \to \mathcal{M}_n(I\!R)$, $i = 0, 1$, are supposed to be smooth with $\mathcal{A}_0(\mathbf{v})$ invertible and $\mathcal{A}_0^{-1}(\mathbf{v})\mathcal{A}_1(\mathbf{v})$ $I\!R$-diagonalizable for all states $\mathbf{v} \in \Omega_\mathbf{v}$. In other words, with fixed $\epsilon > 0$, (3) is nothing but a nonlinear hyperbolic system with viscous perturbation. For simplicity in the discussion, all the fields in the underlying hyperbolic model (*i.e.* obtained formally when setting $\epsilon = 0$ in (3)) are supposed to be genuinely nonlinear. Here, the central assumption is that neither $\mathcal{A}_0$ nor $\mathcal{A}_1$ coincide with Jacobian matrices : namely the nonlinear PDE model (3) does not write in conservation form. Motivated by several models from the Physics, we shall assume the existence of a smooth change of variable $\mathbf{v} \in \Omega_\mathbf{v} \to \mathbf{u}(\mathbf{v}) \in \Omega_\mathbf{u}$ so that the smooth solutions of (3) obey the following equivalent form :

$$\partial_t \mathbf{u}(\mathbf{v}_\epsilon) + \partial_x \mathcal{F}(\mathbf{u}(\mathbf{v}_\epsilon)) = \epsilon \mathcal{R}(\mathbf{u}(\mathbf{v}_\epsilon), \partial_x \mathbf{u}(\mathbf{v}_\epsilon), \partial_{xx} \mathbf{u}(\mathbf{v}_\epsilon)), \quad x \in I\!R,\ t > 0, \tag{4}$$

which we shall write for short :

$$\partial_t \mathbf{u}_\epsilon + \partial_x \mathcal{F}(\mathbf{u}_\epsilon) = \epsilon \mathcal{R}(\mathbf{u}_\epsilon, \partial_x \mathbf{u}_\epsilon, \partial_{xx} \mathbf{u}_\epsilon), \quad x \in I\!R,\ t > 0. \tag{5}$$

Notice of course that in contrast with (3), the regularization term in (5) :

$$\mathcal{R}(\mathbf{u}(\mathbf{v}_\epsilon), \partial_x \mathbf{u}(\mathbf{v}_\epsilon), \partial_{xx} \mathbf{u}(\mathbf{v}_\epsilon)) \equiv \nabla_\mathbf{v} \mathbf{u}(\mathbf{v}_\epsilon) \mathcal{A}_0^{-1}(\mathbf{v}_\epsilon) \partial_x(\mathcal{D}(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon), \tag{6}$$

is in general nonconservative while the underlying first order operator in (5) now stands in conservation form. To shade further light in such a change of variable, let us consider the underlying hyperbolic system in (3) (*i.e.* again setting formally $\epsilon = 0$) so as to evaluate the following scalar product :

$$\nabla_\mathbf{v} u_i(\mathbf{v}) \cdot \partial_t \mathbf{v} + \nabla_\mathbf{v} u_i(\mathbf{v}) \cdot \mathcal{A}_0^{-1}(\mathbf{v})\mathcal{A}_1(\mathbf{v})\partial_x \mathbf{v} = 0, \tag{7}$$

where for any given $i \in \{1, .., n\}$, the smooth function $u_i : \Omega_\mathbf{v} \to I\!R$ denotes the ith component of the vector-valued function $\mathbf{v} \to \mathbf{u}(\mathbf{v})$. Due to (5), the above scalar equation necessarily recasts in conservation form :

$$\partial_t u_i(\mathbf{v}) + \partial_x \mathcal{F}_i(\mathbf{u}(\mathbf{v})) = 0, \tag{8}$$

so that, by definition, the scalar functions u_i, $\mathcal{F}_i$ play the role of an entropy pair (trivial or non trivial) for smooth solutions of the underlying hyperbolic system in (3). Hence, the ith component of the regularization term (6) is nothing but the associated entropy rate of production. At the present stage, we do not assume convexity in the mapping $\mathbf{v} \to u_i(\mathbf{v})$ nor suppose that each of the possible (second order) nonconservative products in (6) keeps a constant sign (say negative). Rephrazing the above observations in (6)–(8), the existence of the change of variable $\mathbf{v} \to \mathbf{u}(\mathbf{v})$ in (5) thus requires the existence of as many additional entropy pairs with independent gradients for (3) than there exist scalar equations involving genuine nonconservative products in (3). Such a requirement might sound rather restrictive but surprisingly, most of the important nonconservative models for complex compressible materials actually achieve it. The interest in the equivalent form (5) over (3) stays in that it allows for a mathematical framework to handle the singular limit $\epsilon \to 0^+$ which turns to be really tractable from the numerical standpoint. The next section introduces such a framework due to Berthon, Coquel, LeFloch [5].

2.1 Definition of weak solutions

In this paragraph, we first address formal issues concerning the singular limit in (3) to then motivate precise definitions that are needed in the forthcoming sections devoted to applications. Since by assumption neither $\mathcal{A}_0$ nor $\mathcal{A}_1$ do coincide with Jacobian matrices, the underlying hyperbolic system in (3), obtained setting $\epsilon = 0$:

$$\mathcal{A}_0(\mathbf{v})\partial_t \mathbf{v} + \mathcal{A}_1(\mathbf{v})\partial_x \mathbf{v} = 0, \quad x \in I\!R,\ t > 0, \tag{9}$$

writes in nonconservative form just like its viscous form (3). Therefore, one cannot formally pass to the limit $\epsilon \to 0^+$ in (3) in the usual sense of distributions, so as to recover (9) in the classical weak sense. But a weak sense is needed since in general, the nonlinear hyperbolic system (9) does not admit smooth solutions : propagating shock waves appear in finite time in smooth initial data. Nevertheless and provided that some suitable estimates on the sequence $\mathbf{v}_\epsilon$ and its derivatives $\partial_t \mathbf{v}_\epsilon$, $\partial_x \mathbf{v}_\epsilon$ are satisfied, one reasonnably expects to get $\epsilon \partial_x \mathcal{D}(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon \rightharpoonup 0, \quad \epsilon \to 0$, together with :

$$\mathcal{A}_0(\mathbf{v}_\epsilon)\partial_t \mathbf{v}_\epsilon \rightharpoonup \mathcal{A}_0(\mathbf{v})\partial_t \mathbf{v}, \quad \mathcal{A}_1(\mathbf{v}_\epsilon)\partial_x \mathbf{v}_\epsilon \rightharpoonup \mathcal{A}_1(\mathbf{v})\partial_x \mathbf{v}, \tag{10}$$

vaguely in the sense of measures so that (9) could be reached in this rather vague sense. The central difficulty stems from the non conservative products in (10) : they involve products of discontinuous functions with measures and thus, they are generally not stable with respect to weak convergence (see [18] and [24] for a definition and counterexamples). At present, several successful and stable definitions exist in the BV framework : we refer the reader to LeFloch [21], [22], Dal Maso, LeFloch, Murat [18] and LeFloch, Tzavaras [24]. These definitions lead to a solution of the Riemann problem in the class of hyperbolic systems with genuine nonlinearity for initial data sufficiently flat [18]. Existence of weak solutions of (9) has been then established on the ground of the random choice method [23]. Weak solutions to the nonconservative system (9) can be thus defined in the class of BV functions. After the classical results by Volpert [29] and Federer [16], such functions can be manipulated as if they were piecewise Lipschitz continuous functions. For the simplicity in this brief review and without significant loss of generality, we restrict from now on attention to piecewise Lipschitz continuous functions.

After [21], [18], and [24], here is now the key issue to be put forward. These suitable theories necessarily come with the property that the singular limits entering (10) intrinsically depends on the sequence $\mathbf{v}_\epsilon$ via the choice of the viscous regularization in (3). The core of this sensitiveness basically finds its root in the non conservative products $\mathcal{A}_0(\mathbf{v})\times\partial_t\mathbf{v}$ and $\mathcal{A}_1(\mathbf{v})\times\partial_x\mathbf{v}$ in (9). Indeed and without reference to the singular limit in (3), such products are ambiguous : already the simplest formal product $H\times\delta$ can be found to be equal to $\alpha\delta$ with α an arbitrary positive constant. Hence, the measures $\mathcal{A}_0(\mathbf{v})\times\partial_t\mathbf{v}$ and $\mathcal{A}_1(\mathbf{v})\times\partial_x\mathbf{v}$ cannot be uniquely defined : this nonuniqueness precisely gives room for the shock solutions to (9) to be regularization dependent. As underlined first by LeFloch [21], uniqueness in the definition of the nonconservative products can be restored with explicit reference to the precise shape of $\mathcal{D}$ in (3).

In sharp contrast is the undependence property met by shock solutions with respect to small scale effects in the setting of conservative (genuinely) nonlinear systems. Indeed assuming $\mathcal{A}_0$ and $\mathcal{A}_1$ to coincide with the jacobian matrices of some flux functions $\mathcal{F}_0$ and $\mathcal{F}_1$, then from suitable estimates on the derivatives of $\mathbf{v}_\epsilon$, $\partial_t\mathcal{F}_0(\mathbf{v}_\epsilon) \rightharpoonup \partial_t\mathcal{F}_0(\mathbf{v})$ and $\partial_x\mathcal{F}_1(\mathbf{v}_\epsilon) \rightharpoonup \partial_x\mathcal{F}_1(\mathbf{v})$ in the usual sense of distributions with $\epsilon\mathcal{D}(\mathbf{v}_\epsilon)\partial_x\mathbf{v}_\epsilon \rightharpoonup 0$, so that we get at points of jump in the limit function $\mathbf{v}$:

$$[-\sigma(\mathcal{F}_0(\mathbf{v}_+)-\mathcal{F}_0(\mathbf{v}_-))+(\mathcal{F}_1(\mathbf{v}_+)-\mathcal{F}_1(\mathbf{v}_-))]\delta_{x-\sigma t}=0, \tag{11}$$

for some speed of propagation σ. These so-called Rankine-Hugoniot jump conditions stay completely free from the particular shape of the viscous matrix $\mathcal{D}$ and allow to define $\mathbf{v}_+ = \mathbf{v}_+(\mathbf{v}_-,\sigma)$ (at least locally) independently of $\mathcal{D}$. As already claimed, such a property cannot hold for nonconservative hyperbolic systems and at points of jump, possible exit states $\mathbf{v}_+$ do depend on $\mathbf{v}_-$ and σ but also deeply on the shape of $\mathcal{D}$: $\mathbf{v}_+ = \mathbf{v}_+(\mathcal{D};\ \mathbf{v}_-,\sigma)$.

Let us now turn considering the equivalent form (5). By contrast to (3), the nonconservative regularization term $\epsilon\mathcal{R}(\mathbf{u}_\epsilon,\partial_x\mathbf{u}_\epsilon,\partial_{xx}\mathbf{u}_\epsilon)$ cannot be expected to converge to zero in the sense of measures as ϵ goes to zero but instead to a (vector-valued) bounded Borel measure $\mu_{\mathbf{u}}\{\mathcal{D}\}$ concentrated on the discontinuities of the limit function $\mathbf{u}$. Such a measure vanishes in the region of continuity of $\mathbf{u}$ and has a non trivial mass, we denote $\mathcal{K}_{\mathcal{D}}(\mathbf{u}_-,\sigma)$, along any curve of discontinuity of $\mathbf{u}$ (see hereafter for the notations). From the very definition (6) of the regularization term $\epsilon\mathcal{R}(\mathbf{u}_\epsilon,\partial_x\mathbf{u}_\epsilon,\partial_{xx}\mathbf{u}_\epsilon)$, it is clear that the mass of $\mu_{\mathbf{u}}\{\mathcal{D}\}$ generically depends on the precise shape of $\mathcal{D}$. The viscosity matrix $\mathcal{D}$ being prescribed in (6), the exit state $\mathbf{u}_+$ at points of jump must then solves the following set of generalized jump relations :

$$-\sigma(\mathbf{u}_+-\mathbf{u}_-)+(\mathcal{F}(\mathbf{u}_+)-\mathcal{F}(\mathbf{u}_-))=\mathcal{K}_{\mathcal{D}}(\mathbf{u}_-,\sigma), \tag{12}$$

hence, another illustration of the inherent small-scale sensitiveness of shock solutions to (9).

Let us now address precise definitions for weak solutions to (9) in the class of piecewise Lipschitz continous functions. These definitions, needed in the forthcoming sections, heavily rely on the properties of travelling solutions to (3). For fixed $\epsilon > 0$, these are smooth solutions to (3), and thus equivalently to the companion system (5), of the form :

$$\left\{ \begin{array}{l} \mathbf{v}_\epsilon(x,t) = \mathbf{w}_\epsilon(x-\sigma t) = \mathbf{w}_\epsilon(\xi), \\ \lim_{\xi\to\pm\infty} \mathbf{w}_\epsilon(\xi) = \mathbf{v}_\pm, \quad \lim_{\xi\to\pm\infty} \frac{d}{d\xi}\mathbf{w}_\epsilon(\xi) = 0, \end{array} \right. \tag{13}$$

where σ denotes the speed of the wave and $\mathbf{v}_-$, $\mathbf{v}_+$ are two states in $\Omega_{\mathbf{v}}$. A solution to (3) of the form (13) must thus solve the following system of ordinary differential equations :

$$(\mathcal{A}_1(\mathbf{w}_\epsilon) - \sigma\mathcal{A}_0(\mathbf{w}_\epsilon))\mathbf{w}'_\epsilon = \epsilon(\mathcal{D}(\mathbf{w}_\epsilon)\mathbf{w}'_\epsilon)', \quad \mathbf{w}'_\epsilon = \frac{d}{d\xi}\mathbf{w}_\epsilon(\xi). \tag{14}$$

Now considering the rescaled function $\mathbf{w} : I\!R \to \Omega_{\mathbf{v}}$ defined by :

$$\mathbf{w}(\frac{\xi}{\epsilon}) = \mathbf{w}_\epsilon(\xi), \tag{15}$$

then $\mathbf{w}$ must solve the next ODE problem free from the parameter ϵ :

$$(\mathcal{A}_1(\mathbf{w}) - \sigma\mathcal{A}_0(\mathbf{w}))\mathbf{w}' = (\mathcal{D}(\mathbf{w})\mathbf{w}')', \tag{16}$$

while achieving the same asymptotic conditions as those stated independently from ϵ in (13). Notice that in the present nonconservative case, (16) cannot be integrated once to give rise to a first order sytem like in the conservative framework. Assuming $\mathcal{D}(\mathbf{w})$ invertible for all $\mathbf{w} \in \Omega_{\mathbf{v}}$, one merely has to consider the extended dynamical system :

$$\left\{ \begin{array}{l} \mathbf{r}' = (\mathcal{A}_1(\mathbf{w}) - \sigma\mathcal{A}_0(\mathbf{w}))\mathcal{D}^{-1}(\mathbf{w})\mathbf{r}, \\ \mathbf{w}' = \mathcal{D}^{-1}(\mathbf{w})\mathbf{r}, \end{array} \right. \tag{17}$$

for which the set of critical points, *i.e.* $(\mathbf{r}, \mathbf{w}) = (0, \mathbf{w})$, is *a priori* unknown.

Remark 1 Several authors have established sufficient conditions on the viscosity matrix $\mathcal{D}$ ensuring the existence of small-amplitude traveling wave solutions to hyperbolic systems with viscous regularization : we refer to the work by Majda, Pego [25] and the references therein. Besides, let us stress that the left state $\mathbf{v}_-$ being fixed, the speed σ has to be properly prescribed so as to meet the Lax compression condition $\lambda_k(\mathbf{v}_-) > \sigma$ in order to give rise to a small-amplitude traveling wave solution (see [25] for the details). We tacitly assume from now on that all the reported conditions on $\mathcal{D}$ and σ are met without further reference.

Let a state $\mathbf{v}_-$ be given and some velocity σ be prescribed so that there exists a critical point $(0, \mathbf{v}_+) = (0, \mathbf{v}_+(\mathbf{v}_-, \sigma))$ that can be reached exponentially fast in the future by a smooth solution $(\mathbf{r}, \mathbf{w})$ of (17) and connecting exponentially fast $(0, \mathbf{v}_-)$ in the past. Notice that generally speaking, $\mathbf{v}_+$ does depend on the fixed viscosity matrix $\mathcal{D}$: namely $\mathbf{v}_+ = \mathbf{v}_+(\mathcal{D}; \mathbf{v}_-, \sigma)$. From (15), we are now in a position to define a sequence of solutions $\{\mathbf{w}_\epsilon\}_{\epsilon>0}$ to (14) with the required asymptotic conditions (13). This sequence obeys $||\mathbf{w}'_\epsilon||_{L^1} = ||\mathbf{w}'||_{L^1} < \infty$ and thus converges strongly in L^1_{loc} to the step function :

$$\mathbf{v}(x,t) = \mathbf{v}_- + (\mathbf{v}_+(\mathcal{D}; \mathbf{v}_-, \sigma) - \mathbf{v}_-)H(x - \sigma t), \quad x \in I\!R, \ t > 0, \tag{18}$$

where H denotes the usual Heaviside function. These considerations have led LeFloch [21] to state :

Definition 2.1 *The limit function (18) is a shock solution to (9), compatible with the viscosity matrix $\mathcal{D}$ in (3).*

To go one step further in the characterization of shock solutions to (9) in the sense of Definition 2.1, notice the next identities, valid for all $\epsilon > 0$:

$$\int_{I\!R_\xi} \mathcal{A}_i(\mathbf{w}_\epsilon(\xi))\mathbf{w}'_\epsilon(\xi)d\xi = \int_{I\!R_\xi} \mathcal{A}_i(\mathbf{w}(\xi))\mathbf{w}'(\xi)d\xi, \quad i = 0, 1, \tag{19}$$

while in view of the asymptotic conditions expressed in (13) :

$$\int_{I\!R_\xi} (\mathcal{D}(\mathbf{w}_\epsilon(\xi))\mathbf{w}'_\epsilon(\xi))'d\xi = 0. \tag{20}$$

Following [21], let us consider an increasing one to one function $\psi : (0,1) \to I\!R$ so as to introduce the following path connecting $\mathbf{v}_-$ to $\mathbf{v}_+$ in the phase space $\Omega_\mathbf{v}$:

$$\phi_\mathcal{D}(s; \mathbf{v}_-, \mathbf{v}_+) = \mathbf{w}(\psi(s)), \quad s \in (0,1). \tag{21}$$

Equipped with these notations, we observe that integrating once the ODE system (14) yields the following set of extended Rankine hugoniot relations :

$$\begin{aligned} -\sigma \int_0^1 \mathcal{A}_0(\phi_\mathcal{D}(s; \mathbf{v}_-, \mathbf{v}_+)) \tfrac{\partial \phi_\mathcal{D}}{\partial s}(s; \mathbf{v}_-, \mathbf{v}_+)ds \\ + \int_0^1 \mathcal{A}_1(\phi_\mathcal{D}(s; \mathbf{v}_-, \mathbf{v}_+)) \tfrac{\partial \phi_\mathcal{D}}{\partial s}(s; \mathbf{v}_-, \mathbf{v}_+)ds = 0, \end{aligned} \tag{22}$$

to be solved by (18). It can be shown that the identity (22) stays invariant by change of parametrization of the path (21). Next, $\mathcal{D}$ being fixed, letting the left state $\mathbf{v}_-$ run in $\Omega_\mathbf{v}$ and the speed σ (suitably) in $I\!R$ give rise (at least locally) to a complete family of travelling waves to (3) and thus to a whole family of shock solutions according to Definition 2.1. In turn, (21) allows to define a family of paths $\phi_\mathcal{D}$ so as to connect left and right states in the shock solutions of (9). This construction provides a particular exemple of the general theory of family of paths introduced by Dal Maso, LeFloch, Murat [18] so as to propose a weakly stable definition of nonconservative products in the BV framework. We shall not enter the details in this review and we refer the reader to [18] for the required material. After [18], a fixed family of path being fixed, the limit system (9) may be written :

$$\Big[\mathcal{A}_0(\mathbf{v})\partial_t\mathbf{v}\Big]_{\phi_{\mathcal{D}}} + \Big[\mathcal{A}_1(\mathbf{v})\partial_x\mathbf{v}\Big]_{\phi_{\mathcal{D}}} = 0, \tag{23}$$

in the sense of the next definition (see [18] for the BV framework) :

Definition 2.2 *A piecewise Lipschitz continuous solution* $\mathbf{v} = \mathbf{v}(x,t)$ *is called a weak solution to (23) iff it satisfies in the strong sense (9) in each region of continuity while at points of jump it obeys (22).*

Some of the forthcoming developments will make use of this definition. Let us next turn considering another relevant definition for weak solutions when addressing the equivalent formulation (5) in the setting of the travelling solutions (13) to (3). Since such solutions are smooth, the sequence of functions $\mathbf{u}_\epsilon = \mathbf{u}(\mathbf{w}_\epsilon)$ equally solve (5) with the asymptotics $\lim_{\xi\to\pm\infty}\mathbf{u}_\epsilon(\xi) = \mathbf{u}_\pm = \mathbf{u}(\mathbf{v}_\pm)$, for all $\epsilon > 0$. Hence and with little abuse in the notations, the next identities hold true for all $\epsilon > 0$:

$$-\sigma\mathbf{u}_\epsilon' + (\mathcal{F}(\mathbf{u}_\epsilon))' = \epsilon\mathcal{R}(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon', \mathbf{u}_\epsilon''), \tag{24}$$

while the rescaled function $\tilde{\mathbf{u}}(\xi) = \mathbf{u}(\mathbf{w})$ defined from (15) satisfies :

$$-\sigma\tilde{\mathbf{u}}' + (\mathcal{F}(\tilde{\mathbf{u}}))' = \mathcal{R}(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}', \tilde{\mathbf{u}}''). \tag{25}$$

Again the sequence $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ is seen to converge strongly in L^1_{loc} to the step function :

$$\mathbf{u}(x,t) = \mathbf{u}_- + (\mathbf{u}_+(\mathcal{D};\mathbf{u}_-,\sigma) - \mathbf{u}_-)H(x-\sigma t), \quad x \in I\!R,\ t > 0. \tag{26}$$

The very interest in the derivation of the limit function (26) stems from its characterization by the following set of generalized Rankine-Hugoniot jump conditions :

$$-\sigma(\mathbf{u}_+ - \mathbf{u}_-) + (\mathcal{F}(\mathbf{u}_+) - \mathcal{F}(u_-)) = \mathcal{K}_{\mathcal{D}}(\mathbf{u}_-, \sigma); \tag{27}$$

where the so-called kinetic function $\mathcal{K}_{\mathcal{D}}(\mathbf{v}_-, \sigma) \in I\!R^n$ is defined thanks to the next identity valid for all $\epsilon > 0$ and derived from (24)–(25) :

$$\epsilon\int_{I\!R_\xi} \mathcal{R}(\mathbf{u}_\epsilon, {\mathbf{u}_\epsilon}', {\mathbf{u}_\epsilon}'')d\xi = \int_{I\!R_\xi} \mathcal{R}(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}', \tilde{\mathbf{u}}'')d\xi \equiv \mathcal{K}_{\mathcal{D}}(\mathbf{u}_-, \sigma). \tag{28}$$

Clearly, the vector-valued kinetic function defined in (28) solely depends on the prescribed state $\mathbf{v}_-$ and velocity σ who gave birth to a travelling wave solution to (5), the viscosity matrix $\mathcal{D}$ being fixed. Equipped with a kinetic-function built from a prescribed viscosity matrix $\mathcal{D}$, Berthon, Coquel, LeFloch [5] introduce the following notion of weak solutions :

Definition 2.3 *Let be given a smooth kinetic function $\mathcal{K}_{\mathcal{D}}$. A piecewise Lipschitz solution $\mathbf{u} = \mathbf{u}(x,t)$ is called a weak solution of the nonconservative limit system in (5) iff in each region of continuity, $\mathbf{u}$ solves in the classical sense :*

$$\partial_t \mathbf{u} + \partial_x \mathcal{F}(\mathbf{u}) = 0, \tag{29}$$

while at points of jump, it obeys the generalized Rankine-Hugoniot conditions (27).

In other words, defining $\mu_{\mathbf{u}}\{\mathcal{D}\}$ the bounded Borel measure which vanishes in the region of continuity of $\mathbf{u}$ and has the mass $\mathcal{K}_{\mathcal{D}}(\mathbf{u}_-, \sigma)$ along any curve of discontinuity of $\mathbf{u}$, Definition 2.3 is equivalent to the requirement that $\mathbf{u}$ solves :

$$\partial_t \mathbf{u} + \partial_x \mathcal{F}(\mathbf{u}) = \mu_{\mathbf{u}}\{\mathcal{D}\}, \quad x \in I\!R, \ t > 0. \tag{30}$$

3 The Euler equations with several independent entropies

This section describes some of the main properties of the following nonconservative system with singular viscous perturbation :

$$\begin{cases} \partial_t \rho^\epsilon + \partial_x \rho u^\epsilon = 0, \\ \partial_t \rho u^\epsilon + \partial_x (\rho u^{\epsilon 2} + \sum_{i=1}^N p_i^\epsilon) = \epsilon \partial_x (\sum_{i=1}^N \mu_i \partial_x u^\epsilon) + \epsilon \partial_x (\sum_{i=1}^N \kappa_i \partial_x T_i^\epsilon), \\ \partial_t \rho \varepsilon_i^\epsilon + \partial_x \rho \varepsilon_i u^\epsilon + p_i^\epsilon \partial_x u^\epsilon = \epsilon \mu_i (\partial_x u^\epsilon)^2 + \epsilon \partial_x (\kappa_i \partial_x T_i^\epsilon), \quad i = 1, ..., N, \end{cases} \tag{31}$$

in the regime of an infinite Reynolds number $\mathcal{R}ey = 1/\epsilon \to +\infty$. Here and with classical notations, $\rho, \rho u$ and $\{\rho \varepsilon_i\}_{i=1,...,N}$ respectively stand for the density, the momentum and N independent internal energies of a complex compressible material. Observe that the system (31) just reads as a natural extension of the usual Navier-Stokes equations when a single pressure is involved in the momentum equation. In (31), N pressure laws enter and are independently governed via N distinct internal energies $\rho\varepsilon_i$. Several models from the physics actually enter the present framework with $N > 1$ and we refer the reader to [1] ,[6] for detailed exemples. The system (31) is given the following condensed form :

$$\partial_t \mathbf{v}_\epsilon + \mathcal{A}(\mathbf{v}_\epsilon) \partial_x \mathbf{v}_\epsilon = \epsilon \mathcal{B}(\mathbf{v}_\epsilon, \partial_x \mathbf{v}_\epsilon, \partial_{xx} \mathbf{v}_\epsilon), \quad x \in I\!R, \ t > 0, \tag{32}$$

which natural phase space reads :

$$\Omega_{\mathbf{v}} = \{\mathbf{v} := (\rho, \rho u, \{\rho \epsilon_i\}_{1 \le i \le N}) \in I\!R^{N+2} / \rho > 0, \ \rho u \in I\!R, \ \rho \epsilon_i > 0, \ 1 \le i \le N\}.$$

At the present stage, (32) does not seem to fit with either the model (3) or (5) we have promoted in the last section. Actually, (32) will be seen hereafter to recast in some instances as (3) but always as (5).

3.1 Closure equations and basic properties

Let us first state the (general) closure equations we assume in (31). The internal energies are assumed to obey the second principle of the thermodynamics, *i.e.* for any given $i \in \{1, ..., N\}$, $\rho\varepsilon_i$ is associated with an entropy ρs_i solution of :

$$-T_i ds_i = d\varepsilon_i + p_i d\tau, \quad \tau = 1/\rho, \tag{33}$$

with the property that the mapping $(\tau, s_i) \to \varepsilon_i(\tau, s_i)$ is strictly convex. Thus we get the required thermodynamic closure equations from (33) :

$$p_i(\tau, s_i) = -\partial_\tau \varepsilon_i(\tau, s_i), \quad T_i(\tau, s_i) = -\partial_{s_i} \varepsilon_i(\tau, s_i),$$

where the temperature $T_i(\mathbf{v})$ is classically assumed to stay positive on $\Omega_{\mathbf{v}}$. As a well known consequence, the well defined mapping $(\tau, \varepsilon_i) \to s_i(\tau, \varepsilon_i)$ is strictly convex and so is also, with little abuse in the notation, the mapping $(\rho, \rho\varepsilon_i) \to \{\rho s_i\}(\rho, \rho\varepsilon_i) := \rho s_i(\frac{1}{\rho}, \frac{\rho\varepsilon_i}{\rho})$. Each pressure law $p_i(\mathbf{v})$ is assumed in addition to obey the general Weyl's conditions for real gases (see [19] for the details). At last, the viscosity laws $\mu_i : \Omega_{\mathbf{v}} \to I\!\!R_+$ and the conductivity laws $\kappa_i : \Omega_{\mathbf{v}} \to I\!\!R_+$, $1 \leq i \leq N$, in (31) are assumed to be smooth non negative functions but with the requirement that for some fixed $\mu_0 > 0$:

$$\sum_{i=1}^{N} \mu_i(\mathbf{v}) > \mu_0, \quad \text{for all } \mathbf{v} \in \Omega_{\mathbf{v}}. \tag{34}$$

All the above assumptions are quite classical within the frame of the usual Navier-Stokes equations (*i.e.* when $N = 1$ in (31)). Owing to these assumptions, our first statement highlights the relationships with this usual setting :

Lemma 3.1 *The underlying first order system in (31), obtained formally setting $\epsilon = 0$:*

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^{N} p_i(\mathbf{v})) = 0, \\ \partial_t \rho\varepsilon_i + \partial_x \rho\varepsilon_i u + p_i(\mathbf{v}) \partial_x u = 0, \quad i = 1, ..., N, \end{cases} \tag{35}$$

is hyperbolic over $\Omega_{\mathbf{v}}$, with the following increasingly arranged eigenvalues :

$$\lambda_1(\mathbf{v}) = u - c < \lambda_{j=2,...,N+1}(\mathbf{v}) = u < \lambda_{N+2}(\mathbf{v}) = u + c, \; c^2(\mathbf{v}) = \sum_{i=1}^{N} c_i^2(\mathbf{v}),$$

where each of the partial sound speed follows from $c_i^2(\mathbf{v}) := (\partial_\rho p_i)_{s_i} > 0$. Under the Weyl's assumption on the pressure laws, the $1-$ and $(N+2)-$ fields are genuinely nonlinear. All the other intermediate fields are linearly degenerate.

The intermediate fields with $i \in \{2, .., N+1\}$ coincide with a contact discontinuity across which the eigenvalue u stays continuous. In other words, discontinuities coming with these fields do not induce ambiguity in all the non conservative products $p_i(\mathbf{v}) \times \partial_x u$ involved in (35). By contrast, the two extreme fields are genuinely nonlinear and are thus responsible for the occurence of shock waves where the velocity u and each of the partial pressures $p_i(\mathbf{v})$ can be seen to achieve non trivial jumps [1], [6]. This is already the case in the standard Euler equations with $N = 1$. Hence and for these extreme discontinuities, ambiguities arise in the nonconservative products entering (35).

3.2 Equivalent formulations

In order to study the singular limit $\epsilon \to 0$ in (31), let us implement the program sketched in Section 2 and thus exhibit all the nontrivial conservation laws (8) satisfied by the smooth solution of (35). The next statement provides such laws but when directly expressed in the presence of the viscous perturbations in (31) :

Proposition 3.2 *Smooth solutions of (31) satisfy the following conservation law :*

$$\partial_t(\rho E)(\mathbf{v}^\epsilon) + \partial_x(\rho H u)(\mathbf{v}^\epsilon) = \epsilon\partial_x((\sum_{i=1}^N \mu_i)u^\epsilon\partial_x u^\epsilon) + \sum_{i=1}^N \partial_x(\kappa_i\partial_x T_i^\epsilon), \tag{36}$$

where the total energy $\rho E : \Omega_\mathbf{v} \to I\!R_+$ and the total enthalpy $\rho H : \Omega_\mathbf{v} \to I\!R_+$ respectively read :

$$(\rho E)(\mathbf{v}) = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^N \rho\varepsilon_i, \quad (\rho H)(\mathbf{v}) = (\rho E)(\mathbf{v}) + \sum_{i=1}^N p_i(\mathbf{v}). \tag{37}$$

These solutions next obey the following N equations :

$$-T_i(\mathbf{v}^\epsilon) \times \{\partial_t(\rho s_i)(\mathbf{v}^\epsilon) + \partial_x(\rho s_i u)(\mathbf{v}^\epsilon)\} = \epsilon\mu_i(\partial_x u^\epsilon)^2 + \epsilon\partial_x(\kappa_i\partial_x T_i^\epsilon), \tag{38}$$

and thus also the N entropy balance equations :

$$\partial_t(\rho s_i)(\mathbf{v}^\epsilon) + \partial_x(\rho s_i u)(\mathbf{v}^\epsilon) = -\epsilon\frac{\mu_i}{T_i^\epsilon}(\partial_x u^\epsilon)^2 - \epsilon\kappa_i\big(\frac{\partial_x T_i^\epsilon}{T_i^\epsilon}\big)^2 - \epsilon\partial_x(\kappa_i\frac{\partial_x T_i^\epsilon}{T_i^\epsilon}). \tag{39}$$

Note from (39) that classical non linear tranformations in the s_i yield further additional balance equations for governing $\varphi(s_1, ..., s_N)$ where $\varphi : \mathbb{R}^N \to \mathbb{R}$ denotes any given arbitrary smooth function. Nevertheless and without specific assumptions on the thermodynamic closure equations (see [1], [6]), none of these additional equations boils down to a non trivial additional conservation law. In the light of this result, we are led to introduce the well-defined change of variable $\mathbf{v} \to \mathbf{u}(\mathbf{v}) = \mathbf{u} = \{\rho, \rho u, \{\rho s_i\}_{1\le i\le N}\}^T$ so as to recast (31) under the form (5) according to :

Proposition 3.3 *Smooth solutions of (31) obey equivalently the system :*

$$\begin{cases} \partial_t \rho^\epsilon + \partial_x \rho u^\epsilon = 0, \\ \partial_t \rho u^\epsilon + \partial_x (\rho u^{\epsilon 2} + \sum_{i=1}^{N} p_i^\epsilon) = \epsilon \partial_x (\sum_{i=1}^{N} \mu_i \partial_x u^\epsilon), \\ \partial_t \rho s_i^\epsilon + \partial_x \rho s_i u^\epsilon = -\epsilon \frac{\mu_i}{T_i^\epsilon} (\partial_x u^\epsilon)^2 - \epsilon \kappa_i \big(\partial_x ln T_i^\epsilon\big)^2 - \epsilon \partial_x \kappa_i \partial_x (ln T_i^\epsilon). \end{cases} \tag{40}$$

Note that the conservation law for the total energy is recovered from (40) as an additional nontrivial law :

$$\partial_t (\rho E)(\mathbf{u}^\epsilon) + \partial_x (\rho H u)(\mathbf{u}^\epsilon) = \epsilon \partial_x ((\sum_{i=1}^{N} \mu_i) u^\epsilon \partial_x u^\epsilon) + \sum_{i=1}^{N} \partial_x (\kappa_i \partial_x T_i^\epsilon). \tag{41}$$

The reason for promoting the equivalent system (40) stems from the important property :

Proposition 3.4 *The function $\mathbf{u} \in \Omega_{\mathbf{u}} \to (\rho E)(\mathbf{u}) \in \mathbb{R}$ is strictly convex.*

We refer to [6] for a proof and related convexity properties. Other relevant equivalent forms are actually available [1], [6] but will not be addressed here for shortness. Let us now illustrate that under specific modelling assumptions, the system (31) takes the form (3). This will help to shade light in the forthcoming numerical methods. Let assume the viscosity laws to be given by N non negative real numbers $\mu_i \in \mathbb{R}+$, $1 \leq i \leq N$, with up to some relabelling $\mu_N > 0$ so that the requirement (34) is met. Then, observe the following $(N-1)$ relations easily derived from (38) :

$$-T_N^\epsilon \frac{\mu_i}{\mu_N} \times \{\partial_t (\rho s_N)^\epsilon + \partial_x (\rho s_N u)^\epsilon\} = \epsilon \mu_i (\partial_x u^\epsilon)^2 + \epsilon \partial_x (\frac{\kappa_N \mu_i}{\mu_N} \partial_x T_N^\epsilon), \tag{42}$$

which substracted from (38) yield the $(N-1)$ additional laws :

$$\begin{array}{r} T_i^\epsilon \times \{\partial_t (\rho s_i)^\epsilon + \partial_x (\rho s_i u)^\epsilon\} - T_N \frac{\mu_i}{\mu_N} \times \{\partial_t (\rho s_N)^\epsilon + \partial_x (\rho s_N u)^\epsilon\} = \\ \epsilon \partial_x (\frac{\kappa_N \mu_i}{\mu_N} \partial_x T_N^\epsilon - \kappa_i \partial_x T_i^\epsilon). \end{array} \tag{43}$$

Equipped with (43), it can be shown (see [6] for a related proof) :

Proposition 3.5 *Let be given N constant non negative viscosity coefficients $\{\mu_i\}_{1 \leq i \leq N}$ with $\mu_N > 0$. Then the smooth solutions of (31) obey equivalently in the $\mathbf{u}$-variable the following system :*

$$\begin{cases} \partial_t \rho^\epsilon + \partial_x \rho u^\epsilon = 0, \\ \partial_t \rho u^\epsilon + \partial_x (\rho u^{\epsilon 2} + \sum_{i=1}^{N} p_i^\epsilon) = \epsilon \partial_x (\sum_{i=1}^{N} \mu_i \partial_x u^\epsilon) + \epsilon \partial_x (\sum_{i=1}^{N} \kappa_i \partial_x T_i^\epsilon), \\ T_i(\mathbf{u}^\epsilon) \times \{\partial_t (\rho s_i)^\epsilon + \partial_x (\rho s_i u)^\epsilon\} - T_N(\mathbf{u}^\epsilon) \frac{\mu_i}{\mu_N} \times \{\partial_t (\rho s_N)^\epsilon + \partial_x (\rho s_N u)^\epsilon\} = \\ \qquad \epsilon \partial_x (\frac{\kappa_N \mu_i}{\mu_N} \partial_x T_N^\epsilon - \kappa_i \partial_x T_i^\epsilon), \\ \partial_t (\rho E)(\mathbf{u}^\epsilon) + \partial_x (\rho H u) = \epsilon \partial_x ((\sum_{i=1}^{N} \mu_i) u^\epsilon \partial_x u^\epsilon) + \sum_{i=1}^{N} \partial_x (\kappa_i \partial_x T_i^\epsilon), \end{cases} \tag{44}$$

which condensed form clearly reads :

$$\mathcal{A}_0(\mathbf{u}_\epsilon)\partial_t\mathbf{u}_\epsilon + \mathcal{A}_1(\mathbf{u}_\epsilon)\partial_x\mathbf{u}_\epsilon = \epsilon\partial_x(\mathcal{D}(\mathbf{u}_\epsilon)\partial_x\mathbf{u}_\epsilon), \quad x \in I\!R,\ t > 0, \tag{45}$$

with $\mathcal{A}_0$ invertible :

$$Det(\mathcal{A}_0(\mathbf{u})) = -\frac{1}{\mu_N}(\sum_{i=1}^{N}\mu_i)(\Pi_{i=1}^{N}T_i(\mathbf{u})) < 0, \quad \textit{for all } \mathbf{u} \in \Omega_{\mathbf{u}}. \tag{46}$$

Remark 2 Observe that for general viscosity laws, the above manipulations generally yield the rather cumbersome form :

$$\mathcal{A}_0(\mathbf{u}_\epsilon)\partial_t\mathbf{u}_\epsilon + \mathcal{A}_1(\mathbf{u}_\epsilon)\partial_x\mathbf{u}_\epsilon = \epsilon\mathcal{B}(\mathbf{u}_\epsilon, \partial_x\mathbf{u}_\epsilon, \partial_{xx}\mathbf{u}_\epsilon), \quad x \in I\!R,\ t > 0, \tag{47}$$

with a nonconservative regularization term. Precisely except when the conductivity laws are set to zero, in which case (44) is still valid but with ratios of viscosities depending on $\mathbf{u}$.

The equivalent form (40) with the additional law (41) stays free from modelling assumptions on both the viscosity and conductivity laws. (40)–(41) will play a central role in the numerical analysis of the system (31) in the singular limit $\epsilon \to 0$ we now address.

3.3 Singular limit

We discuss on the ground of the mathematical tools introduced in Section 2 the limit system obtained from (31) as the rescaling parameter ϵ goes to zero. For reasons put forward in Section 2, the $2N$ viscosity and conductivity laws entering the singular viscous perturbation in (31) are tacitly fixed from now on, except when otherwise specified. With little abuse in the notations, this set of $2N$ constitutive laws is referred hereafter as to the viscous closure $\mathcal{D}$.

Focusing ourselves on the notion of weak solutions in the class of piecewise Lipschitz continuous functions, we first report the main properties of the rescaled travelling wave solutions to (31) with $\epsilon = 1$ (see (15)–(16)) :

$$\mathbf{u}(x - \sigma t) = \mathbf{u}(\xi), \quad \lim_{\xi\to\pm\infty}\mathbf{u}(\xi) = \mathbf{u}_\pm, \quad \lim_{\xi\to\pm\infty}\mathbf{u}'(\xi) = 0, \tag{48}$$

for some speed $\sigma \in I\!R$ and states $\mathbf{u}_-$, $\mathbf{u}_+$ in $\Omega_{\mathbf{u}}$. Under the asymptotic conditions expressed in (48), the above function $\mathbf{u}$ has to solve (40) (again with $\epsilon = 1$) and thus the following $(N+2)\times(N+2)$ ODE system :

$$\begin{cases} -\sigma d_\xi\rho + d_\xi\rho u = 0, \\ -\sigma d_\xi\rho u + d_\xi\rho u^2 + \sum_{i=1}^{N} p_i(\mathbf{v})) = d_\xi(\sum_{i=1}^{N}\mu_i d_\xi u), \\ -\sigma d_\xi\rho s_i + d_\xi\rho s_i u = -\frac{\mu_i}{T_i}(d_\xi u)^2 - \kappa_i\big(\frac{d_\xi T_i}{T_i}\big)^2 - d_\xi(\kappa_i\frac{d_\xi T_i}{T_i}). \end{cases} \tag{49}$$

Classical considerations allow for studying travelling solutions coming solely with the first genuinely nonlinear field. Indeed, travelling solutions for the symmetrical extreme field are just recovered when reversing the sign of the velocity and ξ while exchanging the role of the endpoints $\mathbf{u}_-$ and $\mathbf{u}_+$. Berthon, Coquel [2], [4] have proved the existence in the large of solutions (48) to the dynamical system (49) with the closure equations discussed in Section 3.1 but under the simplifying assumption of zero heat conductivities. Such an assumption is made to allow for a Lasalle invariance principle for the large ODE system (49) which in connexion with a suitable Lyapunov function yield the existence of an endpoint $\mathbf{u}_+ \in \Omega_{\mathbf{u}}$ as soon as the state $\mathbf{u}_-$ and the velocity σ are prescribed according to :

Proposition 3.6 *Let $\mathbf{u}_- \in \Omega_{\mathbf{u}}$ be given. Let us consider a velocity σ subject to the Lax compression condition :*

$$u_- - c(\mathbf{u}_-) > \sigma, \quad c^2(\mathbf{u}_-) = \sum_{i=1}^{N} \partial_\rho p_i(\mathbf{u}_-). \tag{50}$$

Then there exists a travelling wave solution (48) to (49), unique (up to some translation) and connecting some (unique) state $\mathbf{u}_+(\mathbf{u}_-, \sigma)$ at $+\infty$.

We conjecture that such a positive result persists in the case of general conductivity laws. Let us underline that due to the nonconservative nature of the dynamical system (49), barely little is known about the exit state $\mathbf{u}_+(\mathbf{u}_-, \sigma)$ except its existence and the property that its precise form heavily depends on the viscous closure $\mathcal{D}$ through the N ratios of the viscosity laws $\mu_i(\mathbf{u})/\sum_{j=1}^{N} \mu_j(\mathbf{u})$. The state $\mathbf{u}_-$ and the speed σ being fixed according to (50), Berthon, Coquel [2], [4] have proved the existence of a smooth manifold of codimension 2 in $\Omega_{\mathbf{u}}$ uniquely made of critical points in the future for the dynamical system (49) with the property that each of these critical points can be reached at $+\infty$ provided that N ratios of viscosities are suitably prescribed. In the setting of N constants viscosities for N given polytropic gases, Chalons, Coquel [8] have shown how to explicitly determine the exit state $\mathbf{u}_+(\mathcal{D}; \mathbf{u}_-, \sigma)$ when simply solving a (known) scalar nonlinear algebraic equation with coefficients depending on $\mathcal{D}$.

Remark 3 In the case of a general viscous closure $\mathcal{D}$, let us stress that the system (49) can be given a dimensionless form. Then all the possible dimensionless exit states $\tilde{\mathbf{u}}_+(\mathcal{D}; ., .)$ (inferred from Proposition 3.6 with $\mathbf{u}_-$ running in $\Omega_{\mathbf{u}}$ and σ in $I\!R$ according to (50)) entirely depend on a reduced set of dimensionless numbers. Typically, N numbers : $\Omega_{\mathbf{u}} \times I\!R \to [0,1]^N$ are in order :

$$0 \leq \mathcal{M} = \frac{c(\mathbf{u}_-)}{u_- - \sigma} \leq 1, \quad 0 \leq \mathcal{M}_i = \frac{\sqrt{\partial_\rho p_i(\mathbf{u}_-)}}{c(\mathbf{u}_-)} \leq 1, \ i \in 1, .., N-1, \tag{51}$$

where $\mathcal{M}$ denotes the inverse of the usual Mach number while the $\mathcal{M}_i$ can be referred as to thermodynamic Mach numbers (see [8] for related reduced numbers). This makes feasible the numerical tabulation of the dimensionless exit states $\tilde{\mathbf{u}}_+(\mathcal{D}; \{\mathcal{M}_i\}(\mathbf{u}_-), \mathcal{M}(\mathbf{u}_-, \sigma))$ on the compact domain (51).

Next, the sequence $\mathbf{u}_\epsilon : \xi \in I\!R \to \mathbf{u}(\xi/\epsilon) \in \Omega_{\mathbf{u}}$ built from the travelling wave solution (48)–(49) is seen to converge strongly in L^1_{loc} when $\epsilon \to 0$ to a limit step function :

$$\mathbf{u}(x,t) = \mathbf{u}_+ + (\mathbf{u}_+(\mathcal{D},\mathbf{u}_-,\sigma) - \mathbf{u}_-)H(x-\sigma t), \quad x \in I\!R, t > 0. \tag{52}$$

With the chainrule (24)–(27) proposed in Section 2, this limit function, referred as to a shock solution to the singular limit in (40), solves the generalized Rankine-Hugoniot jump conditions :

$$\begin{cases} -\sigma(\rho_+ - \rho_-) + ((\rho u)_+ - (\rho u)_-) = 0, \\ -\sigma((\rho u)_+ - (\rho u)_-) + ((\rho u^2 + \sum_{i=1}^N p_i(\mathbf{u}))_+ - (\rho u^2 + \sum_{i=1}^N p_i(\mathbf{u}))_-) = 0, \\ -\sigma((\rho s_i)_+ - (\rho s_i)_-) + ((\rho s_i u)_+ - (\rho s_i u)_-) = \mathcal{K}\{\mathcal{D}\}_i(\mathbf{u}_-,\sigma), \quad i = 1,..,N, \end{cases} \tag{53}$$

where for each $i \in \{1,...,N\}$, the kinetic functions are given by :

$$\mathcal{K}\{\mathcal{D}\}_i(\mathbf{u}_-,\sigma) = -\int_{I\!R_\xi} \Big\{\frac{\mu_i}{T_i}(d_\xi u)^2 + \kappa_i(\frac{d_\xi T_i}{T_i})^2\Big\}(\mathbf{u}(\xi))d\xi \leq 0. \tag{54}$$

Remark 4 In practical issues, once the right state $\mathbf{u}_+(\mathcal{D};\mathbf{u}_-,\sigma)$ has been numerically solved (see Remark 3), the required kinetic functions are evaluated thanks to the identities (see [8] for instance) :

$$\mathcal{K}\{\mathcal{D}\}_i(\mathbf{u}_-,\sigma) = \rho_-(u_- - \sigma)((s_i)_+(\mathcal{D};\mathbf{u}_-,\sigma) - (s_i)_-), \quad i = 1,..,N, \tag{55}$$

in place of the equivalent but cumbersome form (54). Of course, dimensionnless forms of (55) are again in order.

Observe that the smooth travelling solution (48) also solves the additional conservation law (41) so as a by-product, we get the additional jump condition :

$$-\sigma((\rho E)(\mathbf{u}_+) - (\rho E)(\mathbf{u}_-)) + ((\rho H u)(\mathbf{u}_+) - (\rho H u)(\mathbf{u}_-)) = 0. \tag{56}$$

For forthcoming numerical reasons, it is then crucial to recognize that the next set of generalized Rankine-Hugoniot jump conditions can be built from (53)–(56) according to :

Proposition 3.7 *Assume that up to some relabelling $\mu_N(\mathbf{u}) > 0$ so that $\mathcal{K}\{\mathcal{D}\}_N(\mathbf{u}_-,\sigma) < 0$. Then the shock solution (52) solves in the $\mathbf{u}$ variable :*

$$\begin{cases} -\sigma(\rho_+ - \rho_-) + ((\rho u)_+ - (\rho u)_-) = 0, \\ -\sigma((\rho u)_+ - (\rho u)_-) + ((\rho u^2 + \sum_{i=1}^N p_i(\mathbf{u}))_+ - (\rho u^2 + \sum_{i=1}^N p_i(\mathbf{u}))_-) = 0, \\ \{(\rho s_i)_+ - (\rho s_i)_-) + ((\rho s_i u)_+ - (\rho s_i u)_-\} \\ -\frac{\mathcal{K}\{\mathcal{D}\}_i(\mathbf{u}_-,\sigma)}{\mathcal{K}\{\mathcal{D}\}_N(\mathbf{u}_-,\sigma)}\{(\rho s_N)_+ - (\rho s_N)_-) + ((\rho s_N u)_+ - (\rho s_N u)_-\} = 0, \ i = 1,...,N-1, \\ -\sigma((\rho E)(\mathbf{u}_+) - (\rho E)(\mathbf{u}_-)) + ((\rho H u)(\mathbf{u}_+) - (\rho H u)(\mathbf{u}_-)) = 0. \end{cases} \tag{57}$$

For any given $i \in \{1,...,N\}$, let us define $\mu_{\mathbf{u}}\{\mathcal{D}\}_i$ the non positive bounded Borel measure which vanishes in the region of continuity of $\mathbf{u}$ and has the mass $\mathcal{K}\{\mathcal{D}\}_i(\mathbf{u}_-,\sigma)$ along any curve of discontinuity of $\mathbf{u}$, we introduce :

Definition 3.8 *The singular limit system (40), in the class of piecewise Lipschitz continuous functions, takes the form of the following extended Euler equations :*

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^{N} p_i(\mathbf{u})) = 0, \\ \partial_t \rho s_i + \partial_x \rho s_i u = \mu_{\mathbf{u}} \{\mathcal{D}\}_i, \quad i = 1, .., N. \end{cases} \tag{58}$$

Weak solutions of (58) obeys the additional non trivial conservation laws in the usual weak sense :

$$\partial_t (\rho E)(\mathbf{u}) + \partial_x (\rho H u)(\mathbf{u}) = 0. \tag{59}$$

Here, weak solutions of the extended Euler equations (58) are understood in the sense of Definition 2.3. For forthcoming numerical purposes, it is useful to consider the setting of viscous closures $\mathcal{D}$ with arbitrary conductivity laws but constant viscosities (or arbitrary viscosity laws with null conductivities). In view of the equivalent form (44), weak solutions of (58) equally solve in the sense of Definition 2.2 :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^{N} p_i(\mathbf{u})) = 0, \\ \Big[T_i(\mathbf{u}) \{\partial_t (\rho s_i) + \partial_x (\rho s_i u\}\Big]_{\phi_{\mathcal{D}}} \\ \qquad -\Big[T_N(\mathbf{u}) \frac{\mu_i}{\mu_N} \{\partial_t (\rho s_N) + \partial_x (\rho s_N u)\}\Big]_{\phi_{\mathcal{D}}} = 0, \quad 1 \le i \le N-1, \\ \partial_t (\rho E)(\mathbf{u}) + \partial_x (\rho H u) = 0, \end{cases} \tag{60}$$

where the nonconservative products take the form (22) at points of jump. To go one step further, it can be easily seen that at such points the following identities hold for any given $i \in \{1, .., N-1\}$:

$$\begin{aligned} T_i(\mathbf{u}_-, \sigma) \{ -\sigma((\rho s_i)_+ - (\rho s_i)_-) + ((\rho s_i u)_+ - (\rho s_i u)_-)\} \\ -T_N(\mathbf{u}_-, \sigma) \tfrac{\mu_i}{\mu_N} \{ -\sigma((\rho s_N)_+ - (\rho s_N)_-) + ((\rho s_N u)_+ - (\rho s_N u)_-)\} = 0, \end{aligned} \tag{61}$$

for some averaged temperatures $\{T_i(\mathbf{u}_-, \sigma)\}_{1 \le i \le N}$. These relations can be understood as a particular case of the generalized jump conditions entering (57). Let us conclude this section with the following (global) existence result of solutions to the Riemann problem for the extended Euler Equations (58)

Theorem 3.9 *(Chalons, Coquel [8]). Let be given N constant non negative viscosity coefficients $\{\mu_i\}_{1 \le i \le N}$ with up to some relabelling $\mu_N > 0$ in the setting of N independent pressure laws for polytropic gases. Let be given two states $\mathbf{u}_L$ and $\mathbf{u}_R$ in $\Omega_{\mathbf{u}}$. Then the Cauchy problem (58) with initial data $\mathbf{u}_0(x) = \mathbf{u}_L,\ x < 0,\ \ \mathbf{u}_R,\ x > 0$ has an unique solution away from vacuum.*

We refer to [8] for a precise definition of vacuum. The proof of the above result requires a sharp characterization of the right states $\mathbf{u}_+(\mathcal{D}, \mathbf{u}_-, \sigma)$ coming with travelling wave solutions. Hence the setting under consideration.

4 Riemann Solvers and Kinetic functions

In this section, we address the numerical approximation of the weak solutions of the extended Euler equations (58). The small scale sensitiveness of shock solutions makes this issue particularly challenging. The core of the difficulty indeed stems from the property of shock solutions to be regularization dependent : the artificial dissipation terms induced by numerical methods tend to corrupt the discrete shocks. Our main purpose in this section is to illustrate how to enforce the artificial diffusion in classical numerical methods to mimic the exact dissipation mechanism. Kinetic functions play a central role in the correction procedure we describe hereafter. This procedure intends to keep all the independent discrete rate of entropy dissipation in the exact balance prescribed by the kinetic functions in the generalized jump conditions (57). A deeply related strategy has been first introduced by Berthon, Coquel [3] with $N = 2$ in terms of a local (cell by cell) nonlinear correction procedure, then extended to the general case $N \geq 2$ by Chalons, Coquel [7]. It has recently received several fully explicit versions in the case $N \geq 2$ in the works by Chalons, Coquel [9], [10] and has been successfully extended to problems with two space variables in [6]. For convenience, all these works address the numerical approximation of the equivalent form (60) under the assumption of general viscosity laws with null conductivity coefficients (or say constant viscosity coefficients for arbitrary conductivity laws, see Remark 2). In this review, we extend the correction procedure to the general case, thus tackling directly the formulation (58) of the extended Euler Equations. To avoid unnecessary technical details with approximate Riemann solvers, the extension is performed on the basis of the pure Godunov method. We first motivate the very need to correct this classical solver when pointing out the origin of its failure. Understanding its roots then dictates the procedure. We conclude when highlighting the deep relations in all the existing techniques.

4.1 Origin of the failure

For simplicity in the notations, we restrict ourselves to uniform cartesian discretization of $I\!R_t \times I\!R_x$ defined by a constant time step Δt and a constant space step Δx. Introducing $x_{j+1/2} = (j + 1/2)\Delta x$ with $j \in \mathbb{Z}$ and $t^n = n\Delta t$ with $n \in \mathbb{N}$, the cartesian grids under consideration then read :

$$\cup_{j\in\mathbb{Z},n\in\mathbb{N}}\mathcal{C}_j^n, \quad \mathcal{C}_j^n = [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1}). \tag{62}$$

The approximate solution of the Cauchy problem (58) with $\mathbf{u}_0$ as initial data, we denote $\mathbf{u}_\lambda(x,t)$ with $\lambda = \Delta t/\Delta x$, is classically sought as a piecewise constant function at each time level t^n :

$$\mathbf{u}_\lambda(x,t^n) := \mathbf{u}_j^n, \quad \text{for all } x \in [x_{j-1/2}, x_{j+1/2}), \quad n > 0, \quad j \in \mathbb{Z}, \tag{63}$$

with when $n = 0$:

$$\mathbf{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x)dx, \quad j \in \mathbb{Z}. \tag{64}$$

Assuming the approximate solution $\mathbf{u}_\lambda(x,t^n)$ to be known at a given time $t^n \geq 0$, this one is then defined for $t \in [t^n, t^{n+1})$ as the solution of the Cauchy problem (58) with $\mathbf{u}_\lambda(x,t^n)$ as initial data. Choosing Δt small enough, *i.e.* under the CFL restriction :

$$\lambda \max_{\mathbf{u}} \rho(\nabla_{\mathbf{u}} \mathcal{F}(\mathbf{u})) \leq \frac{1}{2}, \tag{65}$$

where the maximum is taken over all the $\mathbf{u}$ under consideration, $\mathbf{u}_\lambda(x,t)$ with $t \in [t^n, t^{n+1})$ is nothing but the juxtaposition of a sequence of non interacting adjacent Riemann solutions $\mathbf{w}((x - x_{j+1/2})/(t-t^n); \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$, centered at each cell interface $x_{j+1/2}$. Let us then classically consider the L^2-projection of this solution at time t^{n+1-} onto piecewise constant functions :

$$\mathbf{u}_j^{n+1-} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_\lambda(x, t^{n+1-})dx, \quad j \in \mathbb{Z}. \tag{66}$$

Easy calculations based on the Green formula show that the averages (66) reexpress conveniently in the form :

$$\left\{ \begin{array}{l} \rho_j^{n+1-} = \rho_j^n - \lambda \Delta\{\rho u\}(\mathbf{w}(0^+; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n) \\ (\rho u)_j^{n+1-} = (\rho u)_j^n - \lambda\Delta\{\rho u^2 + \sum_{i=N}^{N} p_i\}(\mathbf{w}(0^+; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n), \\ (\rho s_i)_j^{n+1-} = (\rho s_i)_j^n - \lambda\Delta\{\rho s_i u\}_{j+1/2}^n + \lambda\ \mu_{\mathbf{u}}\{\mathcal{D}\}_i(\mathcal{C}_j^n), \end{array} \right. \tag{67}$$

where for any given $i \in \{1, ..., N\}$, $\mu_{\mathbf{u}}\{\mathcal{D}\}_i(\mathcal{C}_j^n)$ denotes the (non positive) mass of the bounded Borel measure $\mu_{\mathbf{u}}\{\mathcal{D}\}_i$ taken over all the possible shock waves that propagate in the cell $\mathcal{C}_j^n$.

According to the usual Godunov's procedure, one would update the approximate solution at time t^{n+1} when defining $\mathbf{u}_\lambda(x, t^{n+1}) = \mathbf{u}_j^{n+1-}$ for all $x \in [x_{j-1/2}, x_{j+1/2})$ and $j \in \mathbb{Z}$. However, such an updating formula would prevent the L^1 norm of the total energy to be preserved with time because of the next statement :

Lemma 4.1 *Under the CFL condition (65), the following inequality holds for all $j \in \mathbb{Z}$:*

$$\begin{aligned} \{\rho E\}(\mathbf{u}_j^{n+1-}) &\leq \tfrac{1}{\Delta x} \textstyle\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(\mathbf{u}_\lambda(x, t^{n+1-}))dx \\ &= \{\rho E\}(\mathbf{u}_j^n) - \lambda\Delta\{\rho H u\}(\mathbf{w}(0^+; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)), \end{aligned} \tag{68}$$

the first inequality being strict generally speaking.

The equality entering the above estimate simply follows from the property that the weak solution $\mathbf{u}_\lambda(x,t)$ with $t \in [t^n, t^{n+1})$ preserves the total energy in view of the additional conservation law (56) valid in $\mathcal{D}'$. Next, the inequality in (68) is just a consequence of the classical Jensen inequality when invoking the strict convexity property of the function $\mathbf{u} \in \Omega_{\mathbf{u}} \to \{\rho E\}(\mathbf{u}) \in \mathbb{R}$ stated in Proposition 3.4. It is well known that in general, the resulting inequality holds strictly. More precisely, it can be seen that (see Coquel, LeFloch [13] for instance) :

$$\begin{aligned} \{\rho E\}(\mathbf{u}_j^{n+1-}) &= \tfrac{1}{\Delta x} \textstyle\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(\mathbf{u}_\lambda(x, t^{n+1-}))dx \\ &\quad -\mathcal{O}(1)\Big(||\mathbf{u}_j^n - \mathbf{u}_{j-1}^n||^2 + ||\mathbf{u}_{j+1}^n - \mathbf{u}_j^n||^2\Big), \end{aligned} \tag{69}$$

for some positive $\mathcal{O}(1)$ depending on the convexity modulus of $\{\rho E\}(\mathbf{u})$. Rephrazing the inequality (68) and its precised form (69), the updating formulae (67) make the L^1-norm of the total energy to dramatically decrease with time as soon as non trivial shock solutions propagate in the discrete solution. Thus the classical Godunov method (67) cannot provide us with a relevant numerical method for approximating the discontinuous solutions under consideration. The estimate (69) is nothing but the origine of the reported failure in the proper capture of shock solutions to (58).

One would be tempted to promote the conservation of the total energy at the discrete level so as to understand (up to some relabelling) the governing equation for $\{\rho s_N\}(\rho, \rho u, \rho E, \{\rho s_i\}_{1\leq i\leq N-1})$ as an additional nontrivial equation. Again and because of convexity properties not reported here, such a strategy can only grossly fail. Indeed, it can be easily shown that this time, the equivalent set of generalized jump conditions (57) cannot hold true at the discrete level (already in the case of a single propagating shock wave). We refer the reader to [7], [9] and [10] for closely related proofs. The correction procedure we now propose finds its root in the negative result we have just reported : we propose to enforce for validity at the discrete level the generalized Rankine-Hugoniot conditions (57).

4.2 Correction procedure

In order to restore the validity of the generalized jump conditions (57) at each time level, we consider a cell by cell procedure to take place as soon as $\mu_{\mathbf{u}}\{\mathcal{D}\}_N(\mathcal{C}_j^n) < 0$. The assumption of a non zero mass in the current cell $\mathcal{C}_j^n$ just expresses the fact that non trivial shock waves do propagate in, so that a correction is needed to counteract the negative effects of (69). We propose to keep unchanged the updated values of both the density and momentum in conservation form :

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad \text{for all } j \in \mathbb{Z}, \tag{70}$$

Next, the N entropies $(\rho s_i)_j^{n+1}$ are sought to be solutions of the following $(N-1)$ relations with $i \in \{1, .., N-1\}$:

$$\begin{aligned} &\left\{(\rho s_i)_j^{n+1} - (\rho s_i)_j^n + \lambda \Delta \{\rho s_i u\}_{j+1/2}^n\right\} - \\ &\tfrac{\mu_{\mathbf{u}}\{\mathcal{D}\}_i(\mathcal{C}_j^n)}{\mu_{\mathbf{u}}\{\mathcal{D}\}_N(\mathcal{C}_j^n)}\left\{(\rho s_N)_j^{n+1} - (\rho s_N)_j^n + \lambda \Delta \{\rho s_N u\}_{j+1/2}^n\right\} = 0, \end{aligned} \tag{71}$$

supplemented by (see (68)) :

$$\{\rho E\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, \{\rho s_i\}_j^{n+1}) \equiv \{\rho E\}_j^n - \lambda \Delta \{\rho H u\}(\mathbf{w}(0^+, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)) \tag{72}$$

Let us underline that the identities (71)–(72) are discrete forms of the N last jump relations in (57). When focusing ourselves in the setting of general viscosity laws with zero conductivity coefficients (or say constant viscosity coefficients with arbitrary conductivity laws), it can be easily seen that for any given $i \in \{1, .., N-1\}$, the ratio $\mu_{\mathbf{u}}\{\mathcal{D}\}_i(\mathcal{C}_j^n)/\mu_{\mathbf{u}}\{\mathcal{D}\}_N(\mathcal{C}_j^n)$ coincides with an averaged form of the ratio $\{\mu_i T_N/\mu_N Ti\}(\mathbf{u})$ in the singular limit in (44) (see Proposition 3.5). In this sense, (71)–(72) just provide a consistent discrete approximation of the limit system in (44). Achieving such a consistency property stays at the basis of the prediction-correction discrete methods developped in [3], [9].

Closely related proofs in these works allow to prove the existence of a unique state $\mathbf{u}_j^{n+1} \in \Omega_{\mathbf{u}}$ solution of (70)–(71)–(72). The discrete method is thus well-defined. Let us stress that an essential ingredient in the proof stays in the property that the total energy has been over-dissipated in the prediction step ! (see [3], [9] for instance for the details).

4.3 Numerical experiments

We present here numerical evidences for illustrating the validity of the numerical strategy that we have proposed in previous section. For that, we consider system (31) where N is taken equal to 3 and the corresponding three internal energies are associated with polytropic ideal gases (thermally and calorically perfect). More precisely, introducing N constant adiabatic exponents $\gamma_i > 1$ for all i=1,...,3, we set

$$\rho\varepsilon_i^\epsilon = \frac{p_i^\epsilon}{\gamma_i - 1}, \quad i = 1, ..., 3.$$

For simplicity we make the choice of constant viscosity laws with a Reynolds number equal to 10^5. As initial data, we propose a step function made of two constant states, called left state and right state in the following, separated by a discontinuity located at $x = 0$ and we approximate the solution on a uniform grid with $\Delta x = 1/300$.

Experiment 1 We set $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.6, 1.8)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$, while the left (l) and right (r) states of initial data read: $(\rho, u, p_1, p_2, p_3)_l = (4., 1., 1.2, 1.4, 1.6)$, $(\rho, u, p_1, p_2, p_3)_r = (2.5568, -1.4305, 0.5162, 0.5103, 0.4999)$.

Experiment 2 We set $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.4, 1.4)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$ Left and right states of initial data now read: $(\rho, u, p_1, p_2, p_3)_l = (3., 1., 1., 1.2, 1.4)$, $(\rho, u, p_1, p_2, p_3)_r = (2.6529, -1.1153, 0.8160, 0.9844, 1.1528)$.

In Figures 1 and 2, we compare some of the corresponding pressure profiles for the exact solutions together with the numerical solutions generated by an usual Godunov approach and our prediction-correction like scheme. As expected, we observe that the classical approach (without correction) fails in capturing the correct solution while the correction step provides us with a numerical solution in good agreement with the exact one. We refer for instance the reader to [6], [9], [10] for additional numerical experiments.

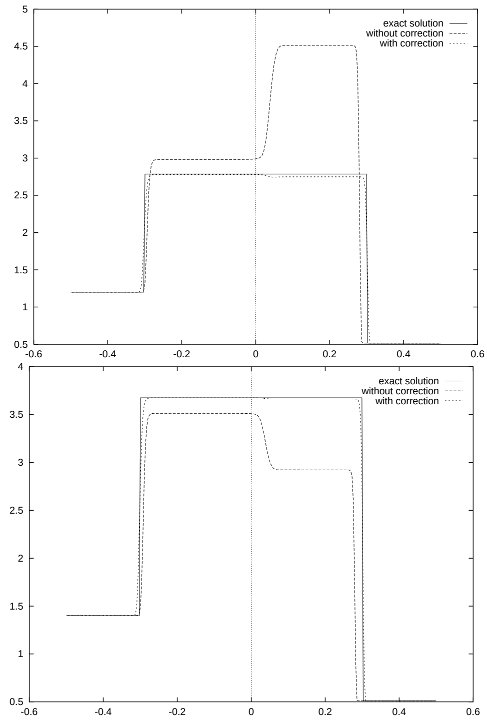

Figure 1: Experiment 1 - Pressures 1 and 2

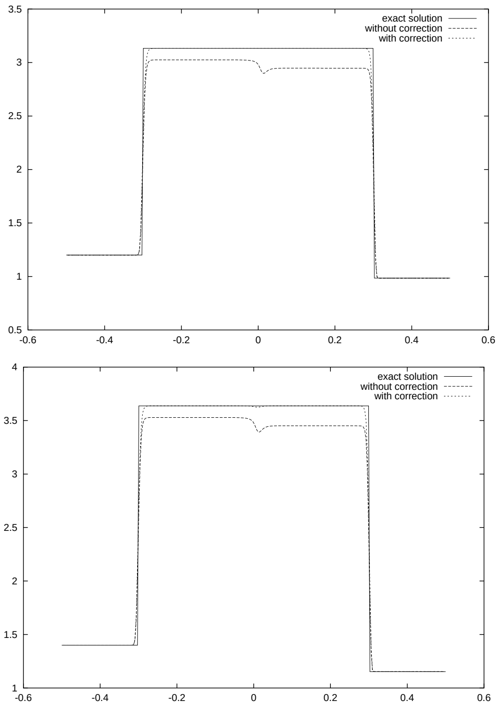

Figure 2: Experiment 2 - Pressures 2 and 3

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