Relaxation and numerical approximation of a two fluid-two pressure model

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ABSTRACT. *This work is concerned with the numerical approximation of a two fluid-two pressure diphasic model. This model can be seen as two Euler systems coupled by a contact discontinuity. On the basis of a relaxation approach, we propose an approximate Riemann solver that captures exactly these contact discontinuities, is conservative for the mass of each phase and the total momentum, and obeys a L^1-stability property.*

KEYWORDS: *Two fluid-two pressure model, relaxation, Riemann solver, finite volumes.*

1. The mathematical model and some basic properties

In recent years, the modelling and computation of two-phase flows by means of a two fluid-two pressure system has gained interest. In this work, we are interested in the numerical approximation of this system which reads, in one space dimension and in the absence of source terms,

$$\left\{\begin{array}{l} \partial_t \alpha_1 + u_I \partial_x \alpha_1 = 0 \, , \\ \partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0 \, , \\ \partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - p_I \partial_x \alpha_1 = 0 \, , \\ \partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0 \, , \\ \partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2 p_2(\rho_2)) - p_I \partial_x \alpha_2 = 0 \, , \end{array}\right. \qquad [1]$$

for $t > 0$ and $x \in \mathbb{R}$. The main unknowns α_k, ρ_k and u_k represent the volume fraction, the density and the velocity of the phase $k = 1, 2$ and we have $\alpha_1 + \alpha_2 = 1$. These are expected to belong to the following phase space

$$\Omega = \{\mathbf{u} = (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2)^t \in \mathbb{R}^5 \text{ such that} \\ \alpha_1 + \alpha_2 = 1,\ \alpha_k > 0 \text{ and } \rho_k > 0 \text{ for } k = 1, 2\} \ .$$

The pressure laws p_k, $k = 1, 2$ are given smooth functions such that $p_k'(\rho_k) > 0$, $\lim_{\rho_k \to 0} p_k(\rho_k) = 0$ and $\lim_{\rho_k \to \infty} p_k(\rho_k) = +\infty$. In order to close this system, we set the interfacial velocity u_I and pressure p_I to be

$$u_I = u_2 \ , \ p_I = p_1(\rho_1) \ . \qquad [2]$$

This choice was first proposed in [BAE 86] and we refer the reader to [GAL 04] for a comprehensive study of more general closure laws. The self-similar solutions of this system are studied in [EMB 92]. The first consequence of [2] is that the void fraction is transported by a pure contact discontinuity. More precisely, it is proved that system [1] admits the following five real eigenvalues

$$\lambda_0 = u_2, \quad \lambda_1 = u_1 - c_1, \quad \lambda_2 = u_1 + c_1, \quad \lambda_3 = u_2 - c_2, \quad \lambda_4 = u_2 + c_2, \quad [3]$$

with a *linearly degenerate* characteristic field associated with λ_0 (the characteristic fields associated with $\{\lambda_i\}_{i=1,\dots,4}$ are genuinely nonlinear). The sound speeds are given by $c_k = \sqrt{p_k'(\rho_k)}$, $k = 1, 2$. It is worth noticing that system [1] is not always hyperbolic. When $u_2 = u_k \pm c_k$, the corresponding right eigenvectors matrix is no longer diagonalizable and the system becomes resonant. This point is known to generate important difficulties like ill-posedness but will not be addressed here. In fact, we are mainly interested in subsonic flows so that the restriction $u_2 \neq u_k \pm c_k$, $k = 1, 2$ will be always satisfied in practice.
Then, the particular choice [2] allows to give sense to the system even if it can not be written in a conservative form, as it was shown in [COQ 02]. Indeed, the linear degeneracy property makes the nonconservative products $u_I \partial_x \alpha_k$ and $p_I \partial_x \alpha_k$ locally well defined by finding four λ_0-Riemann invariants whose gradients are linearly independent. Classical considerations lead to the following parametrisation of the so-called admissible λ_0-contact discontinuities.
Let us note h_1 the enthalpy of the phase 1 defined as $h_1(\rho_1) = e_1(\rho_1) + \frac{p_1(\rho_1)}{\rho_1}$, where the internal energy e_1 verifies $e_1'(\rho_1) = \frac{p_1(\rho_1)}{\rho_1^2}$.

Theorem 1 *Let $\mathbf{u}_-$ and $\mathbf{u}_+$ be two constant states in Ω. These states can be joined by an admissible λ_0-contact discontinuity if and only if the jump relations*

$$\left\{ \begin{array}{c} u_2 := u_{2-} = u_{2+} \ , \\ m := \alpha_{1-}\rho_{1-}(u_{1-} - u_2) = \alpha_{1+}\rho_{1+}(u_{1+} - u_2) \ , \\ m u_{1-} + \alpha_{1-} p_1(\rho_{1-}) + \alpha_{2-} p_2(\rho_{2-}) = m u_{1+} + \alpha_{1+} p_1(\rho_{1+}) + \alpha_{2+} p_2(\rho_{2+}) \ , \\ \dfrac{m^2}{2\alpha_{1-}^2\rho_{1-}^2} + h_1(\rho_{1-}) = \dfrac{m^2}{2\alpha_{1+}^2\rho_{1+}^2} + h_1(\rho_{1+}) \ , \end{array} \right. \qquad [4]$$

hold true. The speed of propagation of the discontinuity is given by u_2.

2. Relaxation approximation

The aim of this section is to propose a suitable relaxation approximation of system [1]. For that, we start from the principle that most of the difficulties arising in the (attempt of) resolution of system [1] are closely related to the nonlinearities of the pressure laws $\rho_k \mapsto p_k(\rho_k)$. In the spirit of [JIN 95], [GOD 00], [CHA 05], we then consider an enlarged system with two additional scalar unknown quantities associated with two linearizations Π_k of the pressure laws p_k. The point is to modify the pressure laws in the convective part of [1] in order to get a quasilinear enlarged system, and to move the nonlinearities in a stiff relaxation term. This relaxation procedure is based on the idea that solutions of the original system [1] are the limit of the solutions of the proposed enlarged system with relaxation source term in the regime of an infinite relaxation coefficient (see for instance [CHA]).
As a relaxation approximation, we propose the following system of balance laws:

$$\left\{\begin{array}{l} \partial_t\alpha_1 + u_I\partial_x\alpha_1 = 0 \ , \\ \partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0 \ , \\ \partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1\Pi_1) - \Pi_I\partial_x\alpha_1 = 0 \ , \\ \partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0 \ , \\ \partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2\Pi_2) - \Pi_I\partial_x\alpha_2 = 0 \ , \\ \partial_t\mathcal{T}_1 + u_I\partial_x\mathcal{T}_1 = \lambda(\tau_1 - \mathcal{T}_1) \ , \\ \partial_t\mathcal{T}_2 + u_I\partial_x\mathcal{T}_2 = \lambda(\tau_2 - \mathcal{T}_2) \ , \end{array}\right. \quad \text{for } t > 0, \ x \in \mathbb{R} \qquad [5]$$

for some positive relaxation parameter $\lambda > 0$. Here, $\mathcal{T}_k$ and Π_k can be understood as relaxation variables associated with the specific volume $\tau_k = 1/\rho_k$ and the pressure p_k, and we set

$$\Pi_k = p_k(1/\mathcal{T}_k) + a_k^2(\mathcal{T}_k - \tau_k) \ , \ k = 1, 2 \ ,$$

for some constants a_k precised just below. These relaxation quantities $\mathcal{T}_k$ and Π_k are expected to converge towards the corresponding equilibrium ones τ_k and p_k in the regime of an infinite relaxation rate λ ($\lambda \to +\infty$). However, for that to be true and in order to prevent the relaxation system [5] from instabilities in the regime of large values of λ ($\lambda \gg 1$), the free parameters a_k must be such that $a_k > \rho_k c_k(\rho_k)$, $k = 1, 2$, for all the ρ_k under consideration. This condition is the so-called Whitham condition (see for instance [CHA 05] and the recent large literature on this subject).
Of course, the proposed interfacial velocity u_I and pressure Π_I are naturally defined by $u_I = u_2, \ \Pi_I = \Pi_1$.
The main interest of this relaxation system lies in the fact that the first order system extracted from [5] admits the following five real eigenvalues

$$\begin{array}{c} \lambda_0^r = u_2 \ , \\ \lambda_1^r = u_1 - a_1\tau_1 \ , \ \lambda_2^r = u_1 + a_1\tau_1 \ , \\ \lambda_3^r = u_2 - a_2\tau_2 \ , \ \lambda_4^r = u_2 + a_2\tau_2 \ , \end{array} \qquad [6]$$

whose associated characteristic fields are *all linearly degenerate* (the speeds λ_k^r, $k = 1, ..., 4$ are nothing but linearizations of λ_k, $k = 1, ..., 4$). In addition, system [5] is hyperbolic as soon as $u_2 \neq \lambda_k^r$, $k = 1, ..., 4$, in perfect analogy with system [1].

3. Numerical approximation

In this section, we describe a relaxation scheme associated with [5], for approximating the weak solutions of [1]. In what follows, Δt and Δx denote the time and space steps and $\nu = \frac{\Delta t}{\Delta x}$. The mesh interfaces and intermediate times are $x_{j+1/2} = j\Delta x$, $j \in \mathbb{Z}$ and $t^n = n\Delta t$, $n \in \mathbb{N}$, respectively. As usual in the context of finite volume methods, $\mathbf{u}_j^n$ represents an approximate value of the solution in the cells $\mathcal{C}_j = [x_{j-1/2}, x_{j+1/2}[$. At time $t = 0$, we set $\mathbf{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x)dx$, $j \in \mathbb{Z}$, where $\mathbf{u}_0$ is a prescribed initial condition.

The relaxation method. The numerical strategy is very classical in the context of relaxation methods and proposes to first treat the convective part of [5], and then to take into account the relaxation source term. We begin by defining some approximate values $\mathbf{v}_j^n$ *at equilibrium* for system [5] when setting $\mathbf{v}_j^n = \left(\mathbf{u}_j^n, (\tau_1)_j^n, (\tau_2)_j^n\right)^t$. The two steps are defined as follows.
Step 1 : Evolution in time ($t^n \rightarrow t^{n+1-}$) In this first step, we solve the first order system extracted from [5], that is with $\lambda = 0$, with initial data $\mathbf{v}(x,0) = \mathbf{v}_j^n$ if $x \in \mathcal{C}_j$, $j \in \mathbb{Z}$. Since this initial condition is piecewise constant, the exact solution is obtained by glueing together the solutions of the Riemann problems set at each interface $x_{j+1/2}$ provided that Δt satisfies the usual CFL condition

$$\frac{\Delta t}{\Delta x}\max_{\mathbf{v}}\{|\lambda_i^r(\mathbf{v})|,\ i = 0, ..., 5\} \leq \frac{1}{2}\ . \qquad [7]$$

More precisely

$$\mathbf{v}_\nu(x,t) = \mathbf{v}_\mathbf{r}(\frac{x - x_{j+1/2}}{t}; \mathbf{v}_j^n, \mathbf{v}_{j+1}^n) \ \text{ for all } \ (x,t) \in]x_j, x_{j+1}[\times[0, \Delta t]\ , \qquad [8]$$

where $(x,t) \mapsto \mathbf{v}_\mathbf{r}(\frac{x}{t}; \mathbf{v}_L, \mathbf{v}_R)$ denotes the self-similar solution of the Riemann problem associated with [5] (with $\lambda = 0$), that is with initial condition

$$\mathbf{v}(x,0) = \begin{cases} \mathbf{v}_L & \text{if} \quad x < 0\ , \\ \mathbf{v}_R & \text{if} \quad x > 0\ . \end{cases} \qquad [9]$$

Moreover, the constants a_k are chosen large enough to verify a *discretized* Whitham condition. Unfortunately, and despite that all the characteristic fields associated with [5] are linearly degenerate, the Riemann solution is difficult to obtain. In particular, it is not given by an explicit formula. Therefore, an approximate solution $(x,t) \mapsto \tilde{\mathbf{v}}_\mathbf{r}(\frac{x}{t}; \mathbf{v}_L, \mathbf{v}_R)$ is going to be used in practice. This point will be discussed in the next paragraph.
We then propose a classical averaging procedure to define the sequence $(\mathbf{v}_j^{n+1-})_j$:

$$\mathbf{v}_j^{n+1-} = \left(\mathbf{u_j}^{n+1-}, (\mathcal{T}_1)_j^{n+1-}, (\mathcal{T}_2)_j^{n+1-}\right)^t. = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{\mathbf{v}}_\nu(x, \Delta t)dx\ . \qquad [10]$$

Step 2 : Relaxation ($t^{n+1-} \rightarrow t^{n+1}$) In this second step, we propose to account for the stiff relaxation source term when solving the ordinary differential equations

$$\partial_t \mathbf{v} = \lambda \mathcal{R}(\mathbf{v})\ ,\ x \in \mathbb{R}\ , \qquad [11]$$

with $\mathcal{R}(\mathbf{v}) = (0,0,0,0,0,\tau_1-\mathcal{T}_1,\tau_2-\mathcal{T}_2)^t$, in the asymptotic regime $\lambda \to \infty$. As initial condition, we take $\mathbf{v}(x,0) = \mathbf{v}_j^{n+1-}$ if $x \in \mathcal{C}_j\ ,\ j \in \mathbb{Z}$. This clearly amounts to set $\mathbf{v}_j^{n+1} = \left(\mathbf{u}_j^{n+1}, (\mathcal{T}_1)_j^{n+1}, (\mathcal{T}_2)_j^{n+1}\right)^t$ with $\mathbf{u}_j^{n+1} = \mathbf{u}_j^{n+1-}$, $(\mathcal{T}_1)_j^{n+1} = (\tau_1)_j^{n+1-}$ and $(\mathcal{T}_2)_j^{n+1} = (\tau_2)_j^{n+1-}$. This completes the description of the method.

The Riemann solution $(x,t) \mapsto \tilde{\mathbf{v}}_{\mathbf{r}}(\frac{x}{t}; \mathbf{v}_L, \mathbf{v}_R)$**.** As mentioned above, the exact Riemann solution of [5] is not explicitly known, except, of course, in the very particular situation $\alpha_{1L} = \alpha_{1R}$ which leads to two decoupled systems for each phase. In the general case, the value of the nonconservative product $\Pi_I \partial_x \alpha_1$ is difficult to calculate and this makes the resolution of the Riemann problem pretty challenging (see [AMB] for more details). Note that this nonconservative product acts on the λ_0-contact discontinuity only. This discontinuity propagates at the speed u_2^* and the Rankine-Hugoniot jump relation applied to the fifth equation in [5] (with $\lambda = 0$) gives

$$-u_2^*[\alpha_2\rho_2 u_2] + [\alpha_2\rho_2 u_2^2 + \alpha_2\Pi_2] + < \Pi_I \partial_x \alpha_1 >= 0\ .$$

Here the jump of a quantity X between the right and left states of a discontinuity is noted $[X]$ and $< \Pi_I \partial_x \alpha_1 >$ refers to the mass of the nonconservative product. Since u_2 is continuous and equals u_2^*, this relation yields

$$< \Pi_I \partial_x \alpha_1 >= -[\alpha_2 \Pi_2]\ .$$

In order to facilitate the resolution of the Riemann problem, we propose not to find the exact value $[\alpha_2\Pi_2]$, *i.e.* not to consider it as an unknown, but to guess it *a priori*. In other words, we replace the actual value $[\alpha_2\Pi_2]$ with an estimation $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)$ depending on the initial states $\mathbf{u}_L$ and $\mathbf{u}_R$. Once this prediction is provided (see below), we are thus led to consider the following system (recall that $\lambda = 0$),

$$\left\{\begin{array}{l} \partial_t \alpha_1 + u_I \partial_x \alpha_1 = 0\ , \\ \partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) = 0\ , \\ \partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1\Pi_1) = -\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)\delta_{x-u_2^* t}\ , \\ \partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) = 0\ , \\ \partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2\Pi_2) = \overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)\delta_{x-u_2^* t}\ , \\ \partial_t \mathcal{T}_1 + u_I \partial_x \mathcal{T}_1 = 0\ , \\ \partial_t \mathcal{T}_2 + u_I \partial_x \mathcal{T}_2 = 0\ , \end{array}\right. \qquad [12]$$

for $t > 0,\ x \in \mathbb{R}$. Solving the Riemann problem associated with [12] turns out to be easier. Indeed, we show in [AMB] that the corresponding self-similar solution may be explicitly obtained after some manipulations on the Rankine-Hugoniot jump relations. The latter are classical on all the waves except for the λ_0-contact discontinuity, since the right-hand side vanishes in [12]. Across the λ_0-wave, they read

$$\left\{\begin{array}{c} u_2^* := u_{2-} = u_{2+}\ , \\ m := \alpha_{1-}\rho_{1-}(u_{1-} - u_2^*) = \alpha_{1+}\rho_{1+}(u_{1+} - u_2^*)\ , \\ m u_{1-} + \alpha_{1-}\Pi_{1-} + \alpha_{2-}\Pi_{2-} = m u_{1+} + \alpha_{1+}\Pi_{1+} + \alpha_{2+}\Pi_{2+}\ , \\ \alpha_{2+}\Pi_{2+} - \alpha_{2-}\Pi_{2-} = \overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)\ . \end{array}\right. \qquad [13]$$

Let us briefly comment this set of relations, in comparison to [4]. First of all, the first three equations are similar. They express the continuity of u_2 and both the mass and momentum conservation. Then, the fourth ones seem to be different but are shown to be equivalent provided that the prediction $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)$ is exact (see below the case of a λ_0-contact discontinuity joining $\mathbf{u}_L$ to $\mathbf{u}_R$).

To sum up, we are able to prove the following accuracy and stability properties of the proposed relaxation method (see [AMB] for the proof).

Theorem 2 (The relaxation method) *The proposed relaxation method :*
(i) (Conservativity) : is always conservative on $\alpha_k\rho_k$, $k = 1, 2$ and $\alpha_1\rho_1u_1+\alpha_2\rho_2u_2$. If, in addition, α_{1j} is constant, the method is also conservative on $\alpha_k\rho_ku_k$, $k = 1, 2$.
(ii) (L^1 stability) : provides numerical solutions that remain in the phase space Ω provided that the free parameters a_k, $k = 1, 2$ are chosen sufficiently large.
(iii) (Isolated λ_0-contact discontinuities) : captures exactly the stationary admissible λ_0-contact discontinuities of the equilibrium system [1]*, provided that $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)$ is chosen as explained below.*

Remark. Due to the averaging procedure [10], it is usual that only the *stationary* admissible λ_0-contact discontinuity may be exactly computed.

To conclude, it remains to precise the way we predict $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)$.

Estimation of $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R)$. Recall that this quantity aims at providing a relevant approximation of the actual jump $[\alpha_2\Pi_2](\mathbf{u}_L, \mathbf{u}_R)$ across the λ_0-contact discontinuity in the Riemann solution of [5]. For the sake of accuracy, we would like this approximate value to be exact when $\mathbf{u}_L$ and $\mathbf{u}_R$ can actually be joined by an admissible λ_0-contact discontinuity, *i.e.* when $\mathbf{u}_- = \mathbf{u}_L$ and $\mathbf{u}_+ = \mathbf{u}_R$ are such that the jump relations [4] are satisfied. With this in mind, we first define a reconstructed subsonic state $\overline{\mathbf{u}}_R$ such that $\overline{\alpha}_{1R} = \alpha_{1R}$ and such that it can be joined to $\mathbf{u}_L$ by an admissible λ_0-contact discontinuity. We compute a first estimate of $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R) = (\alpha_2\Pi_2)(\overline{\mathbf{u}}_R) - (\alpha_2\Pi_2)(\mathbf{u}_L)$. In a symmetric way, we reconstruct a subsonic state $\overline{\mathbf{u}}_L$ and compute a second estimate $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R) = (\alpha_2\Pi_2)(\mathbf{u}_R) - (\alpha_2\Pi_2)(\overline{\mathbf{u}}_L)$. Of course, we note that if $\mathbf{u}_L$ and $\mathbf{u}_R$ are joined by an admissible λ_0-contact discontinuity, we have $\overline{\mathbf{u}}_L = \mathbf{u}_L$ and $\overline{\mathbf{u}}_R = \mathbf{u}_R$ so that both possibilities coincide with the exact jump $[\alpha_2\Pi_2](\mathbf{u}_L, \mathbf{u}_R)$. We give then a criterion to chose among these two, namely we chose the estimate corresponding to the situation which is closer , in the sense of the Euclidean distance, to the original Riemann problem initial data. Actually this criterion doesn't seem to play an important role: in the numerical tests we performed, no difference can be spotted if one estimate is preferred to the other (see Fig 2).

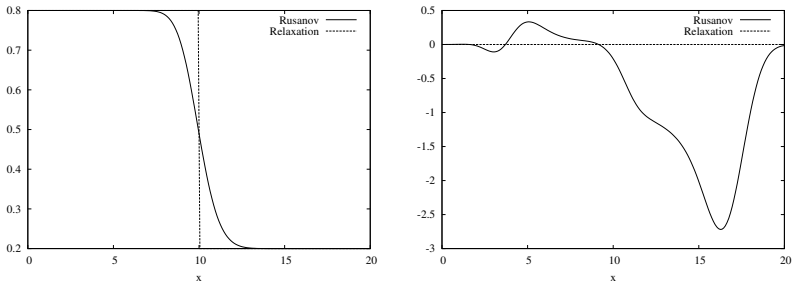
Remark. Notice that if $\alpha_{1L} = \alpha_{1R}$ we have by definition $\overline{\mathbf{u}}_L = \mathbf{u}_R$ and $\overline{\mathbf{u}}_R = \mathbf{u}_L$, and then $\overline{[\alpha_2\Pi_2]}(\mathbf{u}_L, \mathbf{u}_R) = 0$ as expected.

4. Numerical experiments

In this section we present two numerical simulations of the two fluid-two pressure diphasic model. Other numerical experiments can be found in [AMB]. The pressure laws of the phases are given by $p_k(\rho_k) = A_k \rho_k^{\gamma_k}$ $(k = 1, 2)$, where γ_k is the adiabatic coefficient of phase k and A_k a constant depending on the entropy of the gas. In this case, coefficient A_k takes the constant value of 10^5 for $k = 1, 2$ and we choose $\gamma_1 = 1.4$ and $\gamma_2 = 1.2$. Initial conditions correspond to shock tube test cases:

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_L & \text{if } x < 10 \ , \\ \mathbf{u}_R & \text{if } x > 10 \ , \end{cases}$$

In the first simulation, we compare the Riemann solver based on the relaxation approach we propose and a non-conservative version of the Rusanov solver which is explained in detail in [GAL 04] when the left and right states are chosen in a way to verify relations [4] exactly with $u_2 = 0$, $i.e.$ they are separated by a pure stationary λ_0-contact wave. Computations are performed on 200 cells. Results are displayed at final time 0.02s on Figure 1. In agreement with Theorem 2 above, the relaxation solver provides a perfectly sharp discontinuity at $x = 10$m while the Rusanov scheme diffuses the profile. In particular, one can easily notice on Figure 1(b) that only the relaxation scheme exactly preserves the stationary profile of velocity u_2. The second simulation corresponds to a general shock tube test where we compare results obtained using the two different estimates of $\overline{[\alpha_2 \Pi_2]}$ presented above. On fine meshes (400 cells here), no difference can be spotted.
Again, we refer the reader to [AMB] for additional simulations.



(a) Fraction α_1 (-) (b) Velocity u_2 (m/s)

Figure 1. *Pure stationary contact discontinuity*

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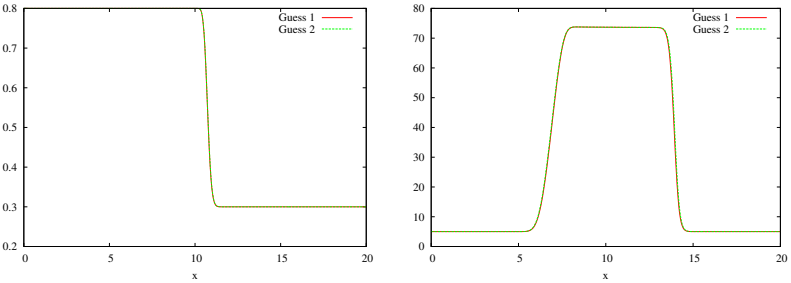

(a) Fraction α_1 (-) (b) Velocity u_2 (m/s)

Figure 2. *Shock tube - Different guess of* $\overline{[\alpha_2 \Pi_2]}$

5. References

[AMB] AMBROSO A., CHALONS C., COQUEL F., GALIÉ T., "Numerical approximation of a two fluid-two pressure model", *in preparation*.

[BAE 86] BAER M. R., NUNZIATO J. W., "A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials", *Int. J. Multiphase Flow*, vol. 12, num. 6, 1986, p. 861-889.

[CHA] CHALONS C., COULOMBEL J., "Relaxation approximation of the Euler equations", *submitted*.

[CHA 05] CHALONS C., COQUEL F., "Navier-Stokes equations with several independent pressure laws and explicit predictor-corrector schemes", *Numer. Math.*, vol. 101, num. 3, 2005, p. 451-478.

[COQ 02] COQUEL F., GALLOUËT T., HÉRARD J.-M., SEGUIN N., "Closure laws for a two-fluid two-pressure model", *C.R. Acad. Sci. Paris, Ser.I*, vol. 334, 2002, p. 927-932.

[EMB 92] EMBID P., BAER M., "Mathematical analysis of a two-phase continuum mixture theory", *Contin. Mech. Thermodyn.*, vol. 4, num. 4, 1992, p. 279-312.

[GAL 04] GALLOUËT T., HÉRARD J.-M., SEGUIN N., "Numerical modeling of two-phase flows using the two-fluid two-pressure approach", *M3AS*, vol. 14, num. 5, 2004, p. 663-700.

[GOD 00] GODLEWSKI E., COQUEL F., IN A., PERTHAME B., RASCLE P., "Some new Godunov methods and relaxation methods for two-phase flow problems", *Godunov methods: theory and applications*, vol. Edited Review. Kluwer Academic/Plenum Publishers, 2000, p. 179-182.

[JIN 95] JIN S., XIN Z., "The relaxation schemes for systems of conservation laws in arbitrary space dimensions", *Comm. Pure Appl. Math.*, vol. 36, 1995, p. 235-276.