Numerical approximation of some hyperbolic problems

C. Chalons

23 juin 2008

Short CV

Academic background

- **1999-2002 :**
  PHD Thesis at ONERA and Ecole Polytechnique
  supervised by P.-G LeFloch and F. Coquel
- **2002-2003 (4 months) :**
  Post-doc position at RWTH Aachen
  supervised by Prof. Ballmann
- **2003-present :**
  Assistant professor at University Paris Diderot-Paris 7
  Member of the J.-L. Lions laboratory
- **2007-2008 :**
  Delegation at CEA Saclay, LETR laboratory

ONERA : French Center for Aerospace Research
CEA : French Center for Nuclear Research

Short CV

Research interests

Theory and Modelling, Numerics and Scientific Computing for

- Hyperbolic systems in conservative and non-conservative form
- Hyperbolic-elliptic systems, nonclassical shocks and phase transitions
- Compressible multi-phase and multi-component flows
- Discretization of sources (*asymptotic-preserving schemes*)
- Finite Volumes schemes (relaxation schemes, averaging versus sampling)
- Coupling of models
- Traffic flows

Outline of the talk

Outline of the talk

Relaxation approximation

The general idea (numerical viewpoint)

Given a PDE system **at equilibrium**

$$\partial_t \mathbf{u} + \nabla.\, \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \omega \subset \mathbb{R}^n, \tag{1}$$

propose a **relaxation** system of the form

$$\partial_t \mathbf{U}^\lambda + \nabla.\, \mathbf{F}(\mathbf{U}^\lambda) = \lambda Q(\mathbf{U}^\lambda), \quad \mathbf{U}^\lambda \in \Omega \subset \mathbb{R}^N, \tag{2}$$

such that

- $\lim_{\lambda \to \infty} \mathbf{U}^\lambda = \mathbf{u}$ ($\lambda > 0$ stands for the relaxation coefficient rate)
- system (2) with $\lambda = 0$ is easier to handle than (1)

Some remarks

- **the litterature is vast on this numerical subject :**
  Jin and Xin '95, Coquel and Perthame '98, Coquel *et al* '01, Bouchut '02...
  see also Natalini, Serre, Yong, Zumbrun...
- the conservative form of (1) is not a restriction
- N may equal $n+1$, $n+2$, $2n$...

Relaxation approximation of the Euler equations

The governing equations

$$\begin{cases} \partial_t \rho + \nabla.\, \rho \mathbf{w} = 0, \qquad t > 0, \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^{\mathbf{d}}, \\ \partial_t \rho \mathbf{w} + \nabla.\, (\rho \mathbf{w} \otimes \mathbf{w} + p(\mathbf{u})\, \mathbf{I_d}) = 0, \\ \partial_t \rho E + \nabla.\, (\rho E + p(\mathbf{u}))\mathbf{w} = 0, \end{cases}$$

The relaxation system

$$\begin{cases} \partial_t \rho + \nabla.\, (\rho \mathbf{w}) = 0, \qquad t > 0, \quad \mathbf{x} \in \mathcal{D}, \\ \partial_t (\rho \mathbf{w}) + \nabla.\, (\rho \mathbf{w} \otimes \mathbf{w} + \Pi\, \mathbf{I_d}) = 0, \\ \partial_t (\rho E) + \nabla.\, ((\rho E + \Pi)\mathbf{w}) = 0, \\ \partial_t (\rho \Pi) + \nabla.\, (\rho \Pi \mathbf{w}) + a^2 \nabla.\, \mathbf{w} = \lambda \rho (p - \Pi), \end{cases}$$

$\longrightarrow$ Recall that $\quad \partial_t \rho p(\mathbf{u}) + \nabla.\, (\rho p(\mathbf{u})\mathbf{w}) + \rho^2 c^2 \nabla.\, \mathbf{w} = 0$

$\longrightarrow$ a is a given real number subject to a stability condition : $a > \rho c(\mathbf{u})$

$\longrightarrow$ **Question : is it possible to prove that** $\lim_{\lambda \to +\infty} \Pi^\lambda = p(\mathbf{u})$?

Convergence result

Theorem

1. The relaxation system admits smooth solutions that converge towards the *local in time* smooth solutions of the Euler equations as $\lambda \to \infty$
2. Arbitrarily large entropic shock of the Euler equations admits a shock profile for the relaxation system

Proof

1. Use the structural properties of the relaxation system, and the main convergence result of Yong '99
2. Pretty technical, use a detailed analysis of the dynamical system satisfied by the shock profile and the center manifold theorem as well

C. Chalons and J.-F. Coulombel
Relaxation approximation of the Euler equations
Analysis and Applications (2008)

NUMERICAL PROCEDURE (TIME EXPLICIT SETTING)

Splitting technique

In order to solve $\partial_t \mathbf{U}^\lambda + \nabla . \, \mathbf{F}(\mathbf{U}^\lambda) = \lambda \mathcal{Q}(\mathbf{U}^\lambda)$

- First, set $\lambda = 0$ and solve $\partial_t \mathbf{U} + \nabla . \, \mathbf{F}(\mathbf{U}) = 0$
- Then, solve $\partial_t \mathbf{U}^\lambda = \lambda \mathcal{Q}(\mathbf{U}^\lambda)$ in the asymptotic regime $\lambda \to \infty$

Key point : all the fields of $\partial_t \mathbf{U} + \nabla . \, \mathbf{F}(\mathbf{U}) = 0$ are **linearly degenerate** (the waves behave as linear waves)

Gathering these two steps leads to a standard finite volume scheme

$$\mathbf{u}_K^{n+1} = \mathbf{u}_K^n - \frac{\Delta t}{|K|} \sum_{e \in \partial K} \mathcal{G}_{e,K}^n |e|$$

- Thanks to the **rotational invariance**

$$\mathcal{G}_{e,K}^n = T_{e,K}^{-1} \, \mathcal{G}(T_{e,K} \mathbf{u}_K^n, T_{e,K} \mathbf{u}_{K_e}^n; i_1)$$

- $\mathcal{G}(.,.; i_1)$ is built from **the Godunov method** in the first space direction i_1

Stability result

Theorem

Under a standard CFL condition and provided that a is sufficiently large ($a > \rho c(\mathbf{u})$), the relaxation scheme

1. keeps the invariant domain : $\rho_K^{n+1} > 0$ and $(\rho e)_K^{n+1} = (\rho E)_K^{n+1} - \dfrac{1}{2}\dfrac{\|(\rho\mathbf{w})_K^{n+1}\|^2}{\rho_K^{n+1}} > 0$
2. satisfies an entropy inequality :
$$(\rho S)(\mathbf{u}_K^{n+1}) - (\rho S)(\mathbf{u}_K^n) + \frac{\Delta t}{|K|}\sum_{e\in\partial K}(\rho S\mathbf{w})_{e,K}^n \leq 0$$
3. obeys a maximum principle :
$$S_K^{n+1} \leq \max_K S_K^n$$
4. is exact for stationary contact discontinuities

C. Chalons and F. Coquel
Navier-Stokes equations with several independent pressure laws and explicit predictor-corrector schemes
Numerisch Math. (2005)

Splitting technique and stationary solutions

Splitting technique

- First, set $\lambda = 0$ and solve $\partial_t \mathbf{U} + \nabla.\, \mathbf{F}(\mathbf{U}) = 0$
- Then, solve $\partial_t \mathbf{U}^\lambda = \lambda Q(\mathbf{U}^\lambda)$ in the asymptotic regime $\lambda \to \infty$

$\longrightarrow$ At least formally, **time convergence** to some steady solution $\mathbf{U}$ would require

$$\nabla.\, \mathbf{F}(\mathbf{U}) = 0 \text{ and } Q(\mathbf{U}) = 0$$

Lemma (convergence failure)

Let $\mathbf{U}$ be such that $\nabla.\, \mathbf{F}(\mathbf{U}) = 0$ and $Q(\mathbf{U}) = 0$ and $a > \rho c$. Then $\mathbf{U}$ also obeys

$$(a^2 - \rho^2 c^2)\, \nabla.\, \mathbf{w} = 0$$

Remark : $\nabla.\, \rho\mathbf{w} = 0$ but $\nabla.\, \mathbf{w} \neq 0$ in general !

Computations of stationary solutions (time-implicit setting)

$\longrightarrow$ We propose, in order **to avoid**

$$\nabla.\, \mathbf{F}(\mathbf{U}) = 0 = \mathcal{Q}(\mathbf{U})$$

and thus **to keep the balance**

$$\nabla.\, \mathbf{F}(\mathbf{U}^\lambda) = \lambda \mathcal{Q}(\mathbf{U}^\lambda)$$

Prediction-correction technique

- First, solve in a **linearized time-implicit** way $\partial_t \mathbf{U}^\lambda + \nabla.\, \mathbf{F}(\mathbf{U}^\lambda) = \lambda \mathcal{Q}(\mathbf{U}^\lambda)$ in the limit $\lambda \to \infty$
- Then, solve **exactly** $\partial_t \mathbf{U}^\lambda = \lambda \mathcal{Q}(\mathbf{U}^\lambda)$ in the limit $\lambda \to \infty$

C. Chalons, F. Coquel and C. Marmignon
Well-balanced time implicit relaxation schemes for the Euler equations
SIAM Journal of Scientific Computing (2008)

Numerical illustrations

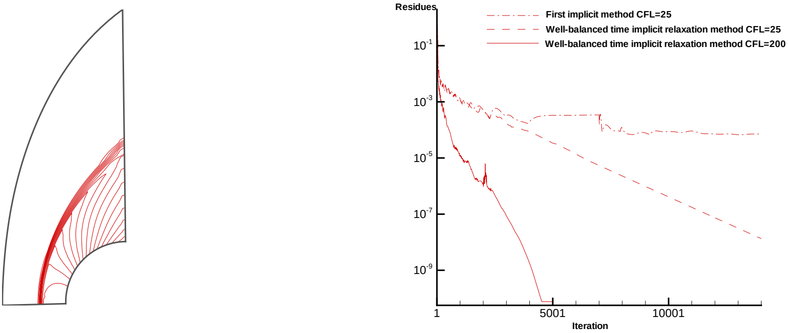


Fig.: ρ (left) and $\| \partial_t \rho \|_{L^2}$ (right)

Some others applications of the relaxation

- **Navier-Stokes equations with several independant pressure laws**
  C. Chalons and F. Coquel
  Multi-pressure Navier-Stokes equations and predictor-corrector schemes
  Numerisch Math. (2005)

  C. Chalons, F. Coquel and C. Marmignon
  Time-implicit approximation of the multi-pressure gas dynamics equations
  to be submitted (2008)

- **Weakly ionized gases**
  C. Chalons, C. Marmignon, O. Rouzaud and T. Soubrié
  Development of a relaxation scheme for weakly ionized gases
  AIAA paper 05-0603 (2005)

- **Two fluid-two pressure diphasic model** (non trivial !)
  A. Ambroso, C. Chalons, F. Coquel and T. Galié
  Relaxation and numerical approximation of a two fluid-two pressure model
  submitted (2008)

Outline of the talk

Context of the work

$\longrightarrow$ Collaboration between

Laboratory Jacques-Louis Lions
CC, F. Coquel, E. Godlewski, F. Lagoutière, P.-A. Raviart, N. Seguin

CEA Saclay
A. Ambroso, J. Segré
B. Boutin and T. Galié (PhD students)

Also participate
S. Kokh (CEA), J.-M. Hérard (EDF R&D)

$\longrightarrow$ We want to solve

$$(S_1) \quad : \quad \partial_t \mathbf{u} + \partial_x \mathbf{f}_-(\mathbf{u}) = 0, \quad x < 0$$

$$(S_2) \quad : \quad \partial_t \mathbf{u} + \partial_x \mathbf{f}_+(\mathbf{u}) = 0, \quad x > 0$$

+ a transmission condition (conservation, continuity...) at $x = 0$

Some works

$\longrightarrow$ Both models (S_1) and (S_2) **share the same physics** (thermohydraulic flow, multiphase flow...) **but are associated with different modelling**

- different closure laws
- different accuracy of description
- different space dimension...

CC, P.-A. Raviart and N. Seguin
The interface coupling of the gas dynamics equations
Quarterly of Applied Mathematics (2008)

A. Ambroso, CC, F. Coquel, E. Godlewski, F. Lagoutière, P.-A. Raviart and N. Seguin
Coupling of general Lagrangian systems
Mathematics of Computation (2008)

A. Ambroso, CC, F. Coquel, E. Godlewski, F. Lagoutière, P.-A. Raviart and N. Seguin
The coupling of homogeneous models for two-phase flows
International Journal of Finite Volumes (2007)

Outline of the talk

GOVERNING EQUATIONS

The model under consideration

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & u(x,t) \in \mathbb{R}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \end{cases}$$

$$+ \qquad \partial_t S(u) + \partial_x Q(u) \leq 0 \qquad (\textbf{single} \text{ entropy inequality})$$

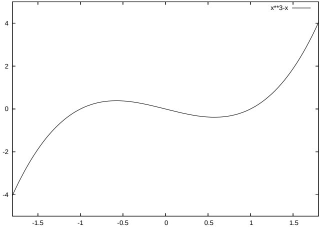


FIG.: Concave-convex flux function

These works are motivated by the computation of phase transitions in van der Waals fluids and nonlinear elastic two-phase material (see for instance my PhD)

Mathematical difficulties

⟶ The Riemann problem admits (up to) *a one-parameter family of solutions*
=> non-uniqueness

⟶ Some of them are both **nonclassical** (they violate the standard Lax shock inequalities) and **physically relevant**

The kinetic relation

⟶ Non-uniqueness can be fixed with an **additional algebraic condition** on each nonclassical shock (σ, u_-, u_+)

⟶ The **kinetic relation** takes the form

$$u_+ = \varphi(u_-) \quad \text{for all nonclassical shocks,} \tag{3}$$

where φ is the kinetic function, and σ is given by the Rankine-Hugoniot relation

See LeFloch '02 monograph

Numerical difficulties

⟶ Nonclassical solutions are **very sensitive w.r.t. small scales and numerical diffusion**

$$u = \lim_{\epsilon \to 0} u^\epsilon \quad \text{with} \quad \partial_t u^\epsilon + \partial_x f(u^\epsilon) = \mathcal{R}(\epsilon \partial_{xx} u^\epsilon, \epsilon^2 \partial_{xxx} u^\epsilon)$$

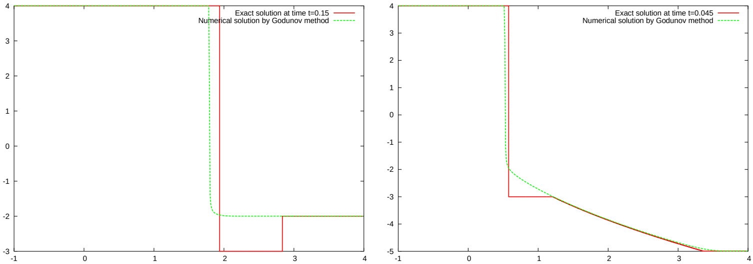


Fig.: Numerical solutions obtained with the Godunov scheme

Numerical difficulties

In the literature, we can distinguish between :

$\longrightarrow$ **the diffuse interface methods :** these methods are *conservative* but work pretty *bad for shocks with large amplitude*
Hayes and LeFLoch '98, LeFloch and Rohde '00...

$\longrightarrow$ **the sharp interface methods :** these methods work *very well*, are *not strictly conservative* (but *convergent*) and *pretty expensive* in general
Hou, Rozakis and LeFloch '99, Merkle and Rohde '06...

Our objective could be...

To get a numerical scheme which :
- is not expensive
- works even for shocks with large amplitude
- provides sharp interfaces
- is conservative

The Transport-Equilibrium strategy

The main idea (2 steps)

⟶ First, at each interface treat the nonclassical shock as a stationary discontinuity
⟶ Then, propagate this discontinuity using a Glimm strategy

- The method is not expensive, works very well, provides sharp interfaces and seems to be convergent
- It is not strictly conservative as the Glimm scheme !

C. Chalons
Transport-Equilibrium Schemes for Computing Nonclassical Shocks
Numerical Methods for Partial Differential Equations (2008)

C. Chalons
Numerical approximation of a macroscopic model of pedestrian flows
SIAM Journal of Scientific Computing (2007)

The reconstruction strategy (see Lagoutière '07, '08)

The main idea

Understand u_j^n as **the projection of a nonclassical discontinuity** related to the Riemann solution associated with initial states $u_l = u_{j-1}^n$ and $u_r = u_{j+1}^n$

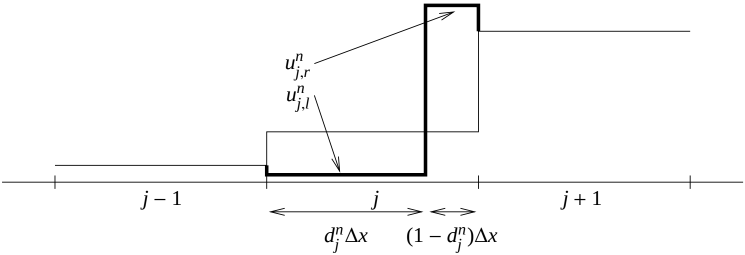


Fig.: A general discontinuous reconstruction with conservation property (the general case).

The reconstruction strategy

Two requirements

$\longrightarrow$ The reconstruction should be **conservative** in the usual sense
$\longrightarrow$ The reconstruction should be **exact for any isolated nonclassical discontinuity**

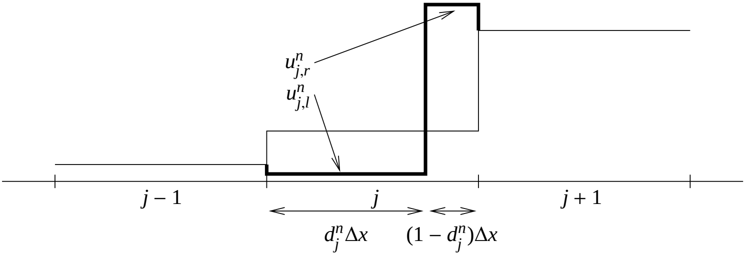


Fig.: A general discontinuous reconstruction with conservation property (the general case).

Stability properties in the case $f' \geq 0$

Theorem

Under a usual CFL restriction and some physical assumptions on f and φ, the scheme

- is **conservative and consistent** in the usual sense of finite volume schemes
- is **exact** on each cell C_j if u_l and u_r are two initial states such that $u_r = \varphi(u_l)$:

$$u_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)dx, \qquad j \in \mathbb{Z}, \ \ n \in \mathbb{N},$$

In particular, the numerical discontinuity **is diffused on one cell** at most

- The method is not expensive, works very well, provides sharp interfaces and is conservative in the usual sense

B. Boutin, C. Chalons, F. Lagoutière and P.-G. LeFloch
A conservative scheme for nonclassical solutions based on kinetic relations
Interfaces and Free Boundaries (2008)

NUMERICAL ILLUSTRATION

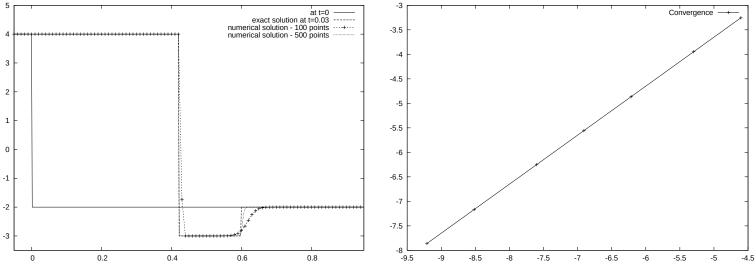


FIG.: Test 2 - Nonclassical and classical shocks – L1 convergence ($\log(E_{L^1})$ versus $\log(\Delta x)$)

Outline of the talk

Several macroscopic models

We denote

ρ : car density

R : maximal density

v : car velocity

- (LWR '56)

$$\partial_t \rho + \partial_x \rho v = 0, \quad v = v(\rho)$$

- (Aw-Rascle '00)

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t \rho w + \partial_x \rho v w = 0 \end{cases}$$

$w = v(\rho, w) + p(\rho)$, $p(\rho) = V_{ref} \ln(\rho/R)$, V_{ref} is a reference velocity

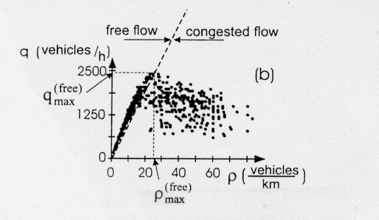
- (Colombo '02)

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t q + \partial_x (qv - Qv) = 0 \end{cases}$$

$v = v(\rho, q)$, q is a weighted momentum and Q a road parameter

A traffic flow model with phase transitions

Experimental data : Kerner '00 (Daimler Benz AG)



Free flow : $(\rho, q) \in \Omega_f$

$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0 \\ q = \rho V \end{cases}$$

$$v_f(\rho) = (1 - \frac{\rho}{R})V$$

Congested flow : $(\rho, q) \in \Omega_c$

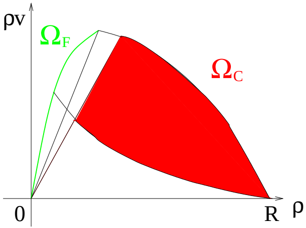
$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0 \\ \partial_t q + \partial_x (q - Q) v = 0 \end{cases}$$

$$v_c(\rho, q) = (1 - \frac{\rho}{R})\frac{q}{\rho}$$

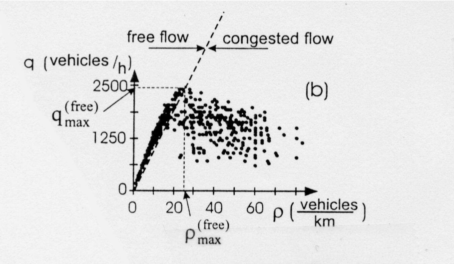
(Colombo '02)

A traffic flow model with phase transitions

Model



Experimental data



$$\Omega_f = \left\{(\rho, q) \in [0, R] \times \mathbb{R}^+ : v_f(\rho) \geq V_f,\ q = \rho \cdot V\right\}$$

$$\Omega_c = \left\{(\rho, q) \in [0, R] \times \mathbb{R}^+ : v_c(\rho, q) \leq V_c,\ \frac{q - Q}{\rho} \in \left[\frac{Q^- - Q}{R}, \frac{Q^+ - Q}{R}\right]\right\}$$

(Colombo ’02)

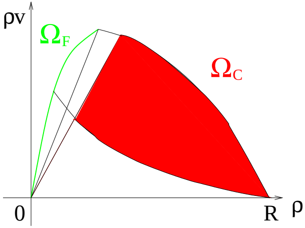
A traffic flow model with phase transitions

- The Riemann problem (Colombo '02) has been solved **uniquely** under some consistency conditions and using **phase transitions** $(\mathbf{u}_-, \mathbf{u}_+, \sigma)$ such that

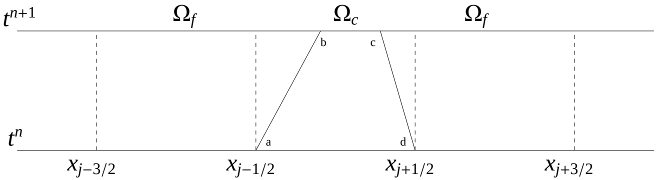
$$\rho v_c(\mathbf{u}_+) - \rho v_f(\mathbf{u}_-) = \sigma\left(\rho(\mathbf{u}_+) - \rho(\mathbf{u}_-)\right)$$

(see Colombo, Goatin and Priuli '06 for the Cauchy problem)

- $\Omega_f \cup \Omega_c$ **is not convex** $\Longrightarrow$ **the Godunov method fails in general**



NUMERICAL APPROXIMATION : A NEW VERSION OF THE GODUNOV SCHEME



Instead of setting

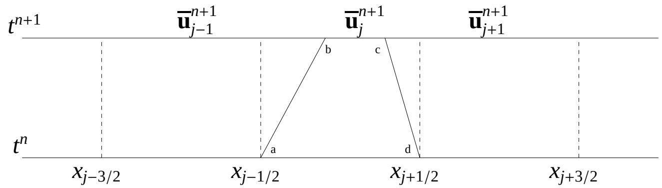
$$\mathbf{u}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, \Delta t) dt$$

we propose

$$\overline{\mathbf{u}}_j^{n+1} = \frac{1}{\overline{\Delta x}_j} \int_{\overline{x}_{j-1/2}}^{\overline{x}_{j+1/2}} \mathbf{v}(x, \Delta t) dt$$

followed by **sampling procedure** to recover the initial mesh and define $\mathbf{u}_j^{n+1}$

Update formulas



Let be given $(a_n)_n$ a **well-distributed random sequence** within $]0, \Delta x[$ (*e.g.* van der Corput sequence)

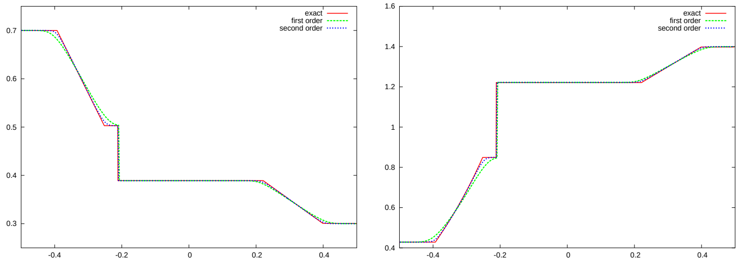
Random choice procedure gives

$$\mathbf{u}_j^{n+1} = \begin{cases} \overline{\mathbf{u}}_{j-1}^{n+1} & \text{if} \quad a_{n+1} \in (0, \frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{+}) \\ \overline{\mathbf{u}}_{j}^{n+1} & \text{if} \quad a_{n+1} \in [\frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{+}, 1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{-}) \\ \overline{\mathbf{u}}_{j+1}^{n+1} & \text{if} \quad a_{n+1} \in [1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{-}, 1) \end{cases}$$

$\sigma_{j+1/2}$ **is the speed of the possible phase transition coming from** $x_{j+1/2}$
$\sigma_{j+1/2}^{+} = \max(\sigma_{j+1/2}, 0)$, $\sigma_{j+1/2}^{-} = \min(\sigma_{j+1/2}, 0)$

Remarks and illustrations

- The algorithm coincides with Godunov's scheme in absence of phase transitions
- The algorithm coincides with Glimm's scheme for an isolated phase transition
- Extension to L^1-stable second-order scheme (MUSCL and RK approaches)



C. Chalons and P. Goatin
Godunov scheme and sampling technique for phase transitions in traffic flows
Interfaces and Free Boundaries (2008)

Numerical approximation of the Aw-Rascle model

The model

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t \rho(v+p) + \partial_x \rho v(v+p) = 0 \end{cases}$$

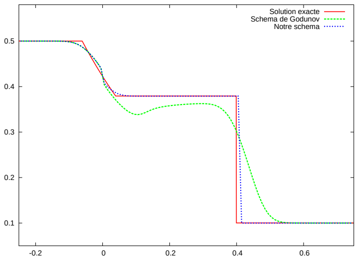
Maximum principle

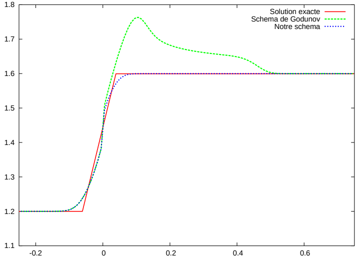
Under an usual CFL condition, the proposed Transport-Equilibrium scheme satisfies the following maximum principles

$$\begin{cases} \inf\limits_{j\in\mathbb{Z}} v_j^0 \leq v_j^n \leq \sup\limits_{j\in\mathbb{Z}} v_j^0\,, \\ \inf\limits_{j\in\mathbb{Z}} (v_j^0 + p(\rho_j^0)) \leq v_j^n + p(\rho_j^n) \leq \sup\limits_{j\in\mathbb{Z}} (v_j^0 + p(\rho_j^0))\,. \end{cases}$$

Remark : these maximum principles are satisfied by a Riemann solution

Numerical approximation of the Aw-Rascle model





C. Chalons and P. Goatin
Transport-Equilibrium schemes for contact discontinuities in traffic flows
Communications in Mathematical Sciences (2007)

Conclusion

- Numerical problems for hyperbolic systems
- Motivated by real applications in fluid mechanics for instance (ONERA, CEA, IFP)
- Construction of new numerical schemes that are handy and able to treat non standard situations
- Which raises sometimes theoretical and scientific computing questions...

ONERA : French Center for Aerospace Research
CEA : French Center for Nuclear Research
IFP : French Center for Oil Research