Chapitre 2

Numerical approximation of the Navier-Stokes equations with several independent specific entropies

Ce chapitre fait l'objet d'un article soumis pour publication : voir [20]

Numerical approximation of the Navier-Stokes equations with several independent specific entropies

C. Chalons [1, 2] and F. Coquel [3]

Résumé

The present work is devoted to the numerical approximation of the solutions of convective-diffusive systems which can be understood as natural extensions of the classical Navier-Stokes (NS) equations. Solutions of these systems are governed by N ($N \geq 2$) independent pressure laws or equivalently N independent specific entropies. Several models from the physics actually enter the present framework. These extended forms of the usual NS equations (i.e. when $N = 1$) are seen to share many properties but with the very difference that, generally speaking, they cannot be recast in full conservation form. This property is known to make the definition of the endpoints of the Travelling Wave (TW) solutions sensitive with respect to the shape of the diffusive tensor. Motivated by the work of Berthon [10] where $N = 2$, here we exhibit $(N-1)$ Generalized Rankine-Hugoniot (GRH) conditions that reflect this sensitivity. We underline the reason why classical algorithms can only fail in satisfying these GRH relations and we show how to enforce them for validity at the discrete level when extending [10] to $N \geq 2$ arbitrary general pressure laws. Several numerical evidences are proposed.

2.1 The mathematical model and some basic properties

In this work, we are interested in the numerical approximation of the solutions of the following convective-diffusive system (in one space dimension for simplicity) :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \qquad x \in I\!R, \quad t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i) = \partial_x ((\sum_{i=1}^N \mu_i) \partial_x u), \\ \partial_t \rho \epsilon_i + \partial_x (\rho \epsilon_i u) + p_i \partial_x u = \mu_i (\partial_x u)^2, \quad i = 1, ..., N, \end{cases} \tag{2.1}$$

where with classical notations, $\rho, \rho u$ and $\{\rho \epsilon_i\}_{i=1,...,N}$ respectively denote the density, the momentum and N independent internal energies. Before addressing the required closure relations, let us stress that (2.1) is nothing but a natural extension of the usual NS equations: namely when considering a single pressure law. Here, N ($N \geq 2$) independent pressure laws $p_i(\rho, \rho\epsilon_i)$ are involved via N independent governing equations for each of the $\rho\epsilon_i$. In (2.1) and for $i = 1, \ldots, N$, μ_i ($\mu_i \geq 0$) denotes a smooth mapping standing for the viscosity law associated with the internal energy $\rho\epsilon_i$. It will be always assumed that $\mu := \sum_{i=1}^N \mu_i > 0$. Depending on the choice of the closure equations for defining each of the μ_i, several models from the physics enter the present framework. Let us quote for instance compressible turbulence models like the $k - \epsilon$ model (see [13]), and more generally multi-scale models (see [19]) for which

1. O.N.E.R.A., B.P. 72, 29 avenue de la Division Leclerc, 92322 Châtillon Cedex, France.
E-mail address: chalons@onera.fr.
2. Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, U.M.R. 7641, École Polytechnique, 91128 Palaiseau Cedex, France.
E-mail address: chalons@cmap.polytechnique.fr.
3. Laboratoire JLL, Boîte courrier 187, 75252 Paris Cedex 05, France.
E-mail address: coquel@ann.jussieu.fr.

a laminar pressure and some turbulent pressures are involved, or models coming from the plasma physics (see [34], [41]).
Now considering the required closure equations for defining each of the N pressure laws, we shall assume that according to the second principle of the thermodynamics:

$$-T_i(\tau,\epsilon_i)ds_i = d\epsilon_i + p_i(\tau,\epsilon_i)d\tau, \quad \tau = 1/\rho, \tag{2.2}$$

defines a strictly convex smooth mapping $(\tau,\epsilon_i) \to s_i(\tau,\epsilon_i) > 0$ for any given $i \in \{1,..,N\}$ where the temperature T_i is classically assumed to stay positive. As a well known consequence, the well defined mapping $(\tau,s_i) \to \varepsilon_i(\tau,s_i)$ is then strictly convex, as well as the mapping $(\rho,\rho\varepsilon_i) \to \{\rho s_i\}(\rho,\rho\varepsilon_i) := \rho s_i(1/\rho,\rho\varepsilon_i/\rho)$ and $(\rho,\rho s_i) \to \{\rho\varepsilon_i\}(\rho,\rho s_i) := \rho\varepsilon_i(1/\rho,\rho s_i/\rho)$ with some little abuse in the notations. Each energy law is assumed to obey in addition the well-known Weyl's assumptions (see [57] for the details). We shall use in particular the following asymptotic conditions:

$$\lim_{\rho s_i\to 0^+}\rho\varepsilon_i(\rho,\rho s_i) = +\infty \text{ and } \lim_{\rho s_i\to+\infty}\rho\varepsilon_i(\rho,\rho s_i) = 0 \ (\textit{i.e.} \lim_{\rho\varepsilon_i\to 0^+}\rho s_i(\rho,\rho\varepsilon_i) = +\infty).$$

Notice that all the above assumptions are quite classical within the frame of the usual NS equations (i.e. with $N=1$). Owing to these assumptions, our first statement highlights the relationships with the usual setting, when defining the following natural phase space for (2.1):

$$\Omega = \{\mathbf{v} := (\rho,\rho u,\{\rho\epsilon_i\}_{1\le i\le N}) \in I\!R^{N+2}/\rho > 0,\ \rho u \in I\!R,\ \rho\epsilon_i > 0,\ 1\le i\le N\}. \tag{2.3}$$

Lemma 8
The first order extracted system from (2.1) is hyperbolic over Ω, with the following increasingly ordered eigenvalues:

$$\lambda_1(\mathbf{v}) = u - c \le \lambda_2(\mathbf{v}) = \ldots = \lambda_{N+1}(\mathbf{v}) = u, \le \lambda_{N+2}(\mathbf{v}) = u + c,\ c^2(\mathbf{v}) = \sum_{i=1}^{N} c_i^2(\mathbf{v}),$$

where each partial sound speed is such that $c_i^2(\mathbf{v}) := (\partial_\rho p_i)_{s_i} > 0$. The $1-$ and $(N+2)-$ fields are genuinely nonlinear, all the other fields being linearly degenerate.
The proof of this easy result is omitted.
The system (2.1) naturally writes in non conservation form. This rises the question of the existence of an admissible change of variables that recasts (2.1) in full conservation form. With this respect, the next claim gives all the additional non trivial equations for the smooth solutions of (2.1) (see [19] or [10] when $N=2$):

Theorem 3
Smooth solutions of (2.1) obey the following conservation law:

$$\partial_t\rho E + \partial_x(\rho E + \sum_{i=1}^{N} p_i)u = \partial_x((\sum_{i=1}^{N}\mu_i)u\partial_x u), \tag{2.4}$$

where the total energy reads $\rho E = (\rho u)^2/2\rho + \sum_{i=1}^{N}\rho\epsilon_i$. In addition, such solutions satisfy:

$$\partial_t\rho s_i + \partial_x\rho s_i u = -\frac{\mu_i}{T_i}(\partial_x u)^2, \quad i = 1,...,N. \tag{2.5}$$

Up to classical nonlinear transformations in the $\{s_i\}_{1\le i\le N}$, these are the only additional non trivial equations (without special assumptions on the $\{\mu_i\}_{1\le i\le N}$).

As a consequence, the system (2.1) cannot be recast generally speaking in full conservation form. However, let us rephrase the above result when investigating suitable change of variables for (2.1). Obviously, the three variables ρ, ρu and ρE are natural candidates since they are governed by conservation laws. Next and for completeness, let us consider without restriction the set $\{\rho s_i\}_{1\leq i\leq N-1}$ to define $\mathbf{u} := (\rho, \rho u, \rho E, \{\rho s_i\}_{1\leq i\leq N-1})$ as an admissible unknown. Then, smooth solutions of (2.1) equivalently satisfy:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \qquad x \in I\!R, \quad t > 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + \sum_{i=1}^N p_i) = \partial_x((\sum_{i=1}^N \mu_i)\partial_x u), \\ \partial_t \rho E + \partial_x(\rho E + \sum_{i=1}^N p_i)u = \partial_x((\sum_{i=1}^N \mu_i) u \partial_x u), \\ \partial_t \rho s_i + \partial_x \rho s_i u = -\frac{\mu_i}{T_i}(\partial_x u)^2, \quad i = 1, ..., N-1. \end{cases} \tag{2.6}$$

Let us notice that the first order underlying system in (2.6) is fully conservative. For conciseness in the forthcoming developments, we right away introduce the following condensed notation for (2.6):

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathcal{R}(\mathbf{u}, \partial_x(\mathcal{B}(\mathbf{u})\partial_x \mathbf{u})).$$

The definition of the tensor $\mathcal{B}$ follows when writing $(\partial_x u)^2 = \partial_x(u\partial_x u) - u\partial_{xx}u$. Now, let us emphasize that whitout further condition, smooth solutions of (2.6) automatically obey with equality the additional PDE:

$$\partial_t \rho s_N(\mathbf{u}) + \partial_x \rho s_N(\mathbf{u}) u = -\frac{\mu_N}{T_N}(\partial_x u)^2. \tag{2.7}$$

Notice that as a consequence of the convexity properties we have put forward at the beginning of this section, the mapping $\mathbf{u} \to \rho s_N(\mathbf{u})$ is strictly convex. Hence, the pair $(\rho s_N, \rho s_N u)$ is nothing but a Lax entropy pair but with a prescribed rate of entropy dissipation. To assess this issue, let us observe after Berthon [10] that despite the identities (2.5) might be understood as N fully independent equations, these do actually evolve proportionally according to:

Corollary 2

Smooth solutions of (2.6) obey:

$$\mu_N T_i\{\partial_t \rho s_i + \partial_x \rho s_i u\} = \mu_i T_N\{\partial_t \rho s_N(\mathbf{u}) + \partial_x \rho s_N(\mathbf{u}) u\}, \quad i = 1, ..., N-1. \tag{2.8}$$

(2.8) expresses nothing but a trivial cancellation property in $(\partial_x u)^2$. However, this observation turns out to be crucial for our numerical purposes. Indeed when applied to TW solutions of (2.6), the identities (2.8) can be understood as $(N-1)$ Generalized Rankine-Hugoniot (GRH) conditions (see [19]). To exemplify this claim, let us temporarily adopt the following closure for the viscosity laws: $\mu_i(\mathbf{u}) = \mu_i^0 T_i$ where for each $i = 1, \ldots, N$, μ_i^0 denotes a given positive real number (see also [10] for complementary examples). Let us recall that such a closure can be inferred from the classical transport theory for dilute gases. Then, (2.8) are easily seen to boil down to:

$$\{\partial_t \rho(\frac{\mu_N}{\mu} s_i - \frac{\mu_i}{\mu} s_N(\mathbf{u})) + \partial_x \rho(\frac{\mu_N}{\mu} s_i - \frac{\mu_i}{\mu} s_N(\mathbf{u}))u\} = 0, \quad i = 1, ..., N-1. \tag{2.9}$$

Therefore, (2.9) clearly provide us with $(N-1)$ jump conditions that, with the three classical Rankine-Hugoniot conditions respectively for ρ, ρu and ρE, yield a complete

set of jump relations for defining the endpoints of a TW solution of (2.6). Let us underline that such endpoints clearly depend on the ratios of the viscosities.
More generally, the proportionality in the evolution in time of all the entropy balance equations (as put forward in (2.8)) expresses a well-known property of general convective-diffusive systems in non conservation form. Namely, the endpoints of their TW solutions entirely depend on the shape of the diffusive tensor under consideration (see [42] and [85]). As underlined in the next section, this key property will make the numerical approximation of the TW solutions of (2.8) a non classical issue (see also [87]). In the present work, we show how to enforce numerically for validity the GRH relations (2.8).

2.2 Numerical approximation

In this section, we present a suitable numerical method for approximating the solutions of (2.1). This method can be understood as a prediction-correction strategy. In the prediction step, we just propose to solve the equivalent form (2.6) with appropriate initial data. Such an approach sounds quite natural but however will be seen to fail in preserving the validity of the GRH conditions at the discrete level. Highlighting the roots of such a failure will naturally motivate the correction step which precisely aims at restoring the validity of (2.8) at each time step.
Let be given a constant time step Δt and a constant space step Δx. Introducing $x_{j+1/2} = (j + 1/2)\Delta x$ for $j \in Z$ and $t^n = n\Delta t$ for $n \in N$, the approximate solution of (2.6), say $\mathbf{u}_\lambda(x, t)$, with $\mathbf{u}_0$ as initial data is classically sought as a piecewise constant function on each slab $C_j^n = [x_{j-1/2}; x_{j+1/2}) \times [t^n; t^{n+1})$ for $(n, j) \in N \times Z$, and we set

$$\mathbf{u}_\lambda(x, t) = \mathbf{u}_j^n \quad \text{for all} \quad (x, t) \in C_j^n, \quad j \in Z, \quad n \in N.$$

Here λ stands for the ratio $\lambda = \Delta t/\Delta x$, and for completeness we set

$$\mathbf{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x)dx.$$

Assuming that the approximate solution $\mathbf{u}_\lambda(x, t^n)$ is known, we propose to advance it to the next time level t^{n+1} according to a classical splitting stategy between the convective and diffusive parts of (2.6). We now describe these two steps into details.

First step ($t^n \to t^{n+1=}$) This step consists in solving the first order underlying system in (2.6), that is to say:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \tag{2.10}$$

with $\mathbf{u}_\lambda(x, t^n)$ as initial data, and supplemented with the following Lax entropy inequality :

$$\partial_t \rho s_N(\mathbf{u}) + \partial_x \rho s_N(\mathbf{u})u \leq 0. \tag{2.11}$$

Let us underline that (2.11) is actually needed for uniqueness of the weak solutions of (2.10). Generically and for any given $\mathbf{u}_L$ and $\mathbf{u}_R$ in the phase space, we then denote by $\mathbf{u}(., \mathbf{u}_L, \mathbf{u}_R)$ the self similar solution of the Riemann problem (2.10)-(2.11) with $\mathbf{u}(x, 0) = \mathbf{u}_L$ if $x < 0$ and $\mathbf{u}(x, 0) = \mathbf{u}_R$ if $x > 0$ as initial data.

To define $\mathbf{u_j^{n+1=}}$, we propose to use the celebrated Godunov method under the usual CFL condition

$$\lambda \max_{\mathbf{u}}(|\lambda_i(\mathbf{u})|, i = 1, ..., N+2) \leq \frac{1}{2}, \tag{2.12}$$

where the maximum is taken over all the $\mathbf{u}$ under consideration. As a consequence, it is well known that $\mathbf{u}_j^{n+1=}$ can be given an updating formula in full conservation form and obeys in addition a discrete cell entropy inequality. More precisely we have:

Theorem 4

Under the CFL condition (2.12), the Godunov method applied to (2.10)-(2.11) with $\mathbf{u}_\lambda(x, t^n)$ as initial data leads to

$$\mathbf{u}_j^{n+1=} = \mathbf{u}_j^n - \lambda \Delta \mathbf{f}_{j+1/2}^n, \quad \Delta \mathbf{f}_{j+1/2}^n = \mathbf{f}_{j+1/2}^n - \mathbf{f}_{j-1/2}^n, \quad j \in Z, \tag{2.13}$$

where we have classically set $\mathbf{f}_{j+1/2}^n = \mathbf{f}(\mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n))$ for all $j \in Z$.
In addition, the following cell entropy inequality holds true:

$$(\rho s_N)_j^{n+1=} := \rho s_N(\mathbf{u}_j^{n+1=}) \leq (\rho s_N)_j^n - \lambda \Delta \{\rho s_N u\}_{j+1/2}^n, \tag{2.14}$$

with $\{\rho s_N u\}_{j+1/2}^n = \{\rho s_N u\}(\mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n))$, as well as the maximum principles on the specific entropies

$$(s_i)_j^{n+1=} \leq \max((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad i = 1, ..., N. \tag{2.15}$$

Let us recall that the inequality (2.14) comes from the entropy selection principle (2.11) together with the Jensen inequality associated with the averaging procedure in the Godunov method. As a consequence, such an inequality is strict generally speaking.

To conclude this first step, observe that one could have used a suitable approximate Riemann solver instead of the exact one for solving the Cauchy problem (2.10)-(2.11). Indeed, the forthcoming analysis entirely stays valid provided that natural stability properties (like consistency, average conservation, average entropy dissipation, ...) are assumed to hold true for the approximate Riemann solver under consideration. We refer the reader to [19] for such considerations. Let us now address the second step.

Second step ($t^{n+1=} \to t^{n+1-}$) This step takes into account the diffusive part of (2.6). It amounts to solve

$$\partial_t \mathbf{u} = \mathcal{R}(\mathbf{u}, \partial_x(\mathcal{B}(\mathbf{u})\partial_x \mathbf{u})), \tag{2.16}$$

with $\mathbf{u}_\lambda(x, t^{n+1=})$ as initial data. Notice that the smooth solutions of this system automatically obey the following additional equation:

$$\partial_t \rho s_N(\mathbf{u}) = -\frac{\mu_N}{T_N}(\partial_x u)^2. \tag{2.17}$$

Several numerical techniques are at hand in the litterature to provide us with an updated solution, say $\mathbf{u}_\lambda(x, t^{n+1-})$, which satisfies (2.16) as well as (2.17) in the sense of the finite differences. Here and after [10], we propose the following implicit discretization, to avoid a too restrictive CFL-like condition:

$$\left\{\begin{array}{l} \rho_j^{n+1-} = \rho_j^{n+1=}, \\ (\rho u)_j^{n+1-} = (\rho u)_j^{n+1=} + \Delta t \overline{\partial_x(\mu \partial_x u)}_j^{n+1-}, \\ (\rho E)_j^{n+1-} = (\rho E)_j^{n+1=} + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \\ (\rho s_i)_j^{n+1-} = (\rho s_i)_j^{n+1=} - \Delta t \overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-}, \quad i = 1, ..., N-1. \end{array}\right. \tag{2.18}$$

Here the finite differences formulae read:

$$\left\{\begin{array}{lll} \overline{\partial_x(\mu\partial_x u)}_j^{n+1-} & = \frac{1}{\Delta x^2}(\mu_{j+1/2}^n(M^n u_{j+1} - M^n u_j) - \mu_{j-1/2}^n(M^n u_j - M^n u_{j-1})), \\ \overline{\partial_x(\mu u\partial_x u)}_j^{n+1-} & = \frac{1}{2\Delta x^2}(\mu_{j+1/2}^n((M^n u_{j+1})^2 - (M^n u_j)^2) - \mu_{j-1/2}^n((M^n u_j)^2 - (M^n u_{j-1})^2)), \\ \overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-} & = \frac{1}{\overline{T_{ij}}^{n+1-}}((\mu_i)_{j+1/2}^n(M^n u_{j+1} - M^n u_j)^2 + (\mu_i)_{j-1/2}^n(M^n u_j - M^n u_{j-1})^2), \end{array}\right.$$

where $M^n X = (X^{n+1-} + X^{n+1=})/2$ and for all i from 1 to N:

$$\overline{T_{ij}}^{n+1-} = \left\{\begin{array}{ll} -\frac{\varepsilon_i(\tau_j^{n+1-},(s_i)_j^{n+1-}) - \varepsilon_i(\tau_j^{n+1-},(s_i)_j^{n+1=})}{(s_i)_j^{n+1-} - (s_i)_j^{n+1=}} & \text{if} \quad (s_i)_j^{n+1-} \neq (s_i)_j^{n+1=}, \\ T_i(\tau_j^{n+1-}, (s_i)_j^{n+1-}) & \text{otherwise.} \end{array}\right.$$

Let us emphasize that since the density is kept unchanged, solving (2.18) just amounts to solve a linear problem in $M^n u$, with a positive definite symmetric matrix. The unknowns $(\rho E)_j^{n+1-}$ and $\{(\rho s_i)_j^{n+1-}\}_{i=1,\dots,N-1}$ then follow with explicit formulae. We do not discuss this step into details and refer the reader to [10] for more informations as well as for the proof of the following statement.

Lemma 9

The updating formulae (2.18) lead to

$$(\rho s_N)_j^{n+1-} := \{\rho s_N\}(\mathbf{u}_j^{n+1-}) = (\rho s_N)_j^{n+1=} - \Delta t \overline{\frac{\mu_N}{T_N}(\partial_x u)^2}_j^{n+1-}. \tag{2.19}$$

Moreover, the following maximum principles hold true:

$$(s_i)_j^{n+1-} \leq (s_i)_j^{n+1=} \leq \max((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad i = 1, ..., N. \tag{2.20}$$

Thus, at the end of this second step, (2.16) and (2.17) can be given a consistent definition in the sense of finite differences. To summarize the prediction step (*i.e.* the first step and the second step), we have with clear notations

$$\begin{array}{rcl} \rho_j^{n+1-} & = & \rho_j^n - \lambda\Delta\{\rho u\}_{j+1/2}^n \\ (\rho u)_j^{n+1-} & = & (\rho u)_j^n - \lambda\Delta\{\rho u^2 + \sum_{i=1}^N p_i\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu\partial_x u)}_j^{n+1-} \\ (\rho E)_j^{n+1-} & = & (\rho E)_j^n - \lambda\Delta\{(\rho u^2 + \sum_{i=1}^N p_i)u\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-} \\ (\rho s_i)_j^{n+1-} & = & (\rho s_i)_j^n - \lambda\Delta\{\rho s_i u\}_{j+1/2}^n - \Delta t\overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-}, \quad i = 1, ..., N-1, \end{array}$$

$$(\rho s_N)_j^{n+1-} \leq (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}_{j+1/2}^n - \Delta t\overline{\frac{\mu_N}{T_N}(\partial_x u)^2}_j^{n+1-}, \tag{2.21}$$

where as a consequence of (2.14), the inequality is strict generally speaking. The classical approach would consist in choosing $\mathbf{u}_j^{n+1} = \mathbf{u}_j^{n+1-}$. But, because of the strict inequality in (2.21) the cancellation property, responsible for (2.8), is lost and the GRH conditions can no longer be satisfied. Obviously, such a failure persists when using a suitable approximate Riemann solver for (2.10)–(2.11). In the next section, numerical evidences will show that the corresponding approximate solutions grossly disagree with the exact ones. This discussion then clearly motivates the need for a correction step to define the updated solution $\mathbf{u}^h(x, t^{n+1})$ so that the required GRH conditions hold. We propose a correction step which extends the procedure introduced in [10] to arbitrary $N \geq 2$ general pressure laws and we prove its relevance.

Correction step ($t^{n+1-} \to t^{n+1}$) We keep unchanged the updated conservative variables, i.e. :

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad (\rho E)_j^{n+1} = (\rho E)_j^{n+1-},$$

and we propose to define the N entropies $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ as the solutions of the following non linear algebraic problem :

$$\begin{gathered} \overline{\mu_i}\overline{T_N}_j^{n+1-}(\{\rho s_N\}(\mathbf{u}_j^{n+1}) - (\rho s_N)_j^n + \lambda\Delta\{\rho s_N u\}_{j+1/2}^n) \\ = \\ \overline{\mu_N}\overline{T_i}_j^{n+1-}((\rho s_i)_j^{n+1} - (\rho s_i)_j^n + \lambda\Delta\{\rho s_i u\}_{j+1/2}^n), \\ i = 1, ..., N-1. \end{gathered} \tag{2.22}$$

Here $\overline{\mu_i}$ is given a natural consistent definition for all $i = 1, ..., N$. Notice that (2.22) is nothing but a discrete version of the GRH conditions (2.8). The numerical fluxes and the definitions of $\overline{T_i}_j^{n+1-}$ entering (2.22) are directly inherited from the prediction step. The following result assesses the relevance of the correction step.

Theorem 5
The algebraic problem (2.22) admits a single solution $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ which obeys in addition the following entropy inequalities

$$\begin{aligned} &(\rho s_i)_j^{n+1} \leq (\rho s_i)_j^n - \lambda\Delta\{\rho s_i u\}_{j+1/2}^n, \quad i = 1, ..., N-1, \\ &(\rho s_N)_j^{n+1} := \{\rho s_N\}(\mathbf{u}_j^{n+1}) \leq (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}_{j+1/2}^n, \end{aligned} \tag{2.23}$$

as well as the maximum principles on the specific entropies

$$(s_i)_j^{n+1} \leq \max((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad i = 1, ..., N, \quad j \in Z. \tag{2.24}$$

Proof
Let us introduce the following convenient notations

$$(\rho s_N)_j^\star = (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}_{j+1/2}^n (\neq \{\rho s_N\}(\mathbf{u}_j^{n+1=})),$$

$$\Gamma_{m,n} = \overline{\mu_m}\overline{T_n}_j^{n+1-} / \overline{\mu_n}\overline{T_m}_j^{n+1-} \geq 0, \quad m = 1, ..., N, \quad , n = 1, ..., N.$$

Let us observe that these quantities are explicitly known at this stage. The first step of the proof consists in recasting (2.22) as follows:

$$\begin{gathered} (\rho s_i)_j^{n+1} = \rho s_i((\rho s_{N-1})_j^{n+1}) = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}((\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}), \\ i = 1, ..., N-2, \\[1em] \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{\rho s_i((\rho s_{N-1})_j^{n+1})\}_{i=1,...,N-2}, (\rho s_{N-1})_j^{n+1}) - (\rho s_N)_j^\star \\ = \\ \Gamma_{N,N-1}((\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}). \end{gathered} \tag{2.25}$$

These identities express the property that the unknowns $(\rho s_i)_j^{n+1}$ for $i = 1, ..., N-2$ can be parametrized by $(\rho s_{N-1})_j^{n+1}$ so that the algebraic problem (2.22) can be solved with $(\rho s_{N-1})_j^{n+1}$ as the only scalar unknown. Thus, existence and uniqueness of the $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ as the solutions of (2.25) is equivalent to prove the existence of a non positive real number $\mathcal{M}_j \leq 0$ (namely $\mathcal{M}_j = (\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}$) such that

$$(\rho s_i)_j^{n+1} = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j, \quad i = 1, ..., N-1,$$

$$f(\mathcal{M}_j) = \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{(\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j\}_{i=1,...,N-1}) -$$
$$\Gamma_{N,N-1}\mathcal{M}_j - (\rho s_N)_j^{\star} = 0. \tag{2.26}$$

Now, since $\partial_{\rho s_i}\{\rho s_N\} = -T_i/T_N < 0$, the mapping $\mathbf{u} \to \rho s_N(\mathbf{u})$ is strictly decreasing with respect to each of the entropies $\{\rho s_i\}_{i=1,...,N-1}$, so that the mapping $\mathcal{M} \to f(\mathcal{M})$ is also strictly decreasing thanks to the positivity of $\Gamma_{m,n}$. Arguing of the asymptotic Weyl's assumptions, let us prove that $\{\rho s_N\}(\mathcal{M}_j) \to +\infty$ (*i.e.* $f(\mathcal{M}_j) \to +\infty$) for a first sufficiently large negative value of $\mathcal{M}_j$, say $\overline{\mathcal{M}_j}$. In that aim, notice that

$$\rho s_i(\mathcal{M}_j) := (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j$$

approaches 0^+ for negative enough values of $\mathcal{M}_j$ for each $i \in \{1, ..., N-1\}$. By the asymptotic condition

$$\lim_{\rho s_i \to 0^+} \rho\varepsilon_i(\rho, \rho s_i) = +\infty,$$

there exists necessarily a first value of $\mathcal{M}_j$, namely $\overline{\mathcal{M}_j}$, solution of

$$(\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} = \sum_{i=1}^{N-1} \rho\varepsilon_i(\rho_j^{n+1}, \rho s_i(\mathcal{M}_j)),$$

that is to say that $\rho\varepsilon_N(\mathcal{M}_j) = 0$ (note that $\rho\varepsilon_N(0) = \rho\varepsilon_N(\rho_j^{n+1=}, (\rho s_N)_j^{n+1=}) > 0$.) But the asymptotic condition

$$\lim_{\rho\varepsilon_N \to 0^+} \rho s_N(\rho, \rho\varepsilon_N) = +\infty$$

just says that

$$\lim_{\mathcal{M}_j \to \overline{\mathcal{M}_j}^+} \rho s_N(\mathcal{M}_j) = +\infty.$$

Note that we have proved in passing that necessarily $\rho s_i(\mathcal{M}_j) > 0$ for all $\overline{\mathcal{M}_j} < \mathcal{M}_j \leq 0$ (since otherwise $f(\mathcal{M}_j) = +\infty$).
Let us next observe the estimate:

$$\begin{aligned} f(0) &= \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) - (\rho s_N)_j^{\star} \\ &= \{\rho s_N\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, (\rho E)_j^{n+1-}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) - (\rho s_N)_j^{\star} \leq 0, \end{aligned}$$

as an immediate consequence of (2.21): indeed, notice that thanks to (2.20) and the decreasing property of $\{\rho s_N\}$ with respect to ρs_i, the following inequality holds:

$$\{\rho s_N\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, (\rho E)_j^{n+1-}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) \leq \{\rho s_N\}(\mathbf{u}_j^{n+1-}).$$

Since $f(0) \leq 0$ with $\lim_{\mathcal{M}_j \to \overline{\mathcal{M}_j}^+} f(\mathcal{M}_j) = +\infty$, we necessarily get from the strict monotonicity of the continuous function f the existence of an unique $\mathcal{M}_j$, solution of (2.26), with the property that

$$\overline{\mathcal{M}_j} < \mathcal{M}_j \leq 0,$$

so that necessarily :

$$(\rho s_i)_j^{n+1} = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j > 0, \quad i = 1, ..., N-1.$$

Turning now establishing the maximum principles (2.24), the entropy inequalities we have just obtained rewrite $(\rho s_i)_j^{n+1} \leq (\rho s_i)_j^{n+1=}$ that is to say $(s_i)_j^{n+1} \leq (s_i)_j^{n+1=}$ for all $i = 1, ..., N-1$ since $\rho_j^{n+1} = \rho_j^{n+1=}$. Invoking now (2.15), we obtain the required conclusion for $i = 1, ..., N-1$. Concerning now ρs_N, we have just proved $(\rho s_N)_j^{n+1} \leq (\rho s_N)_j^{\star}$ (but $(s_N)_j^{n+1} \geq (s_N)_j^{n+1=}$), which is classically sufficient to obtain the required maximum principle in the Godunov method setting. This concludes the proof.

2.3 Numerical experiments

In this section, we give some numerical evidences to illustrate the relevance of the prediction-correction scheme we have proposed. To that purpose, we deal with N internal energies of polytropic ideal gases (thermally and calorically perfect) associated with N constant adiabatic exponents $\gamma_i > 1$ for all i=1,...,N. Hence, the total energy reads $\rho E = (\rho u)^2/2\rho + \sum_{i=1}^N p_i/(\gamma_i - 1)$. For simplicity we consider constant viscosity laws with a Reynolds number equal to 10^5.
We consider as initial data two constant states, called left state and right state in the following, separated by a discontinuity at $x = 0$ and we approximate the solution on a uniform grid with $\Delta x = 1/300$.

Experiment 1 We set $N = 3$, $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.6, 1.8)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$, while the left and right states of the initial data read: $(\rho, u, p_1, p_2, p_3)_{left} = (4., 1., 1.2, 1.4, 1.6)$, $(\rho, u, p_1, p_2, p_3)_{right} = (2.5568, -1.4305, 0.5162, 0.5103, 0.4999)$.

Experiment 2 We set $N = 3$, $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.4, 1.4)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$ while the left and right states of the initial data read: $(\rho, u, p_1, p_2, p_3)_{left} = (3., 1., 1., 1.2, 1.4)$, $(\rho, u, p_1, p_2, p_3)_{right} = (2.6529, -1.1153, 0.8160, 0.9844, 1.1528)$.

In the Figures 2.1 and 2.2, we compare some of the corresponding pressure profiles for the exact solutions (see [19] for the derivation of exact solutions) together with the numerical solutions generated by the scheme both with and without the correction step. As expected, we observe that the classical approach (without correction) fails in capturing the correct solution while the correction step provides us with a numerical solution in good agreement with the exact one. We refer the reader to [19] for additional numerical experiments.

To conclude this paper, let us emphasize that several distinct methods have been actually developed and analysed (see [19]). Their design principle is always focused at preserving the validity of the GRH conditions we have put forward in (2.8). In

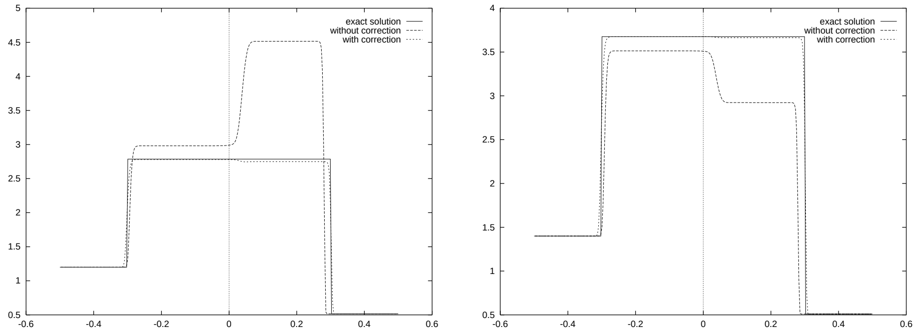

FIG. 2.1 – *Experiment 1 - Pressures 1 and 2*

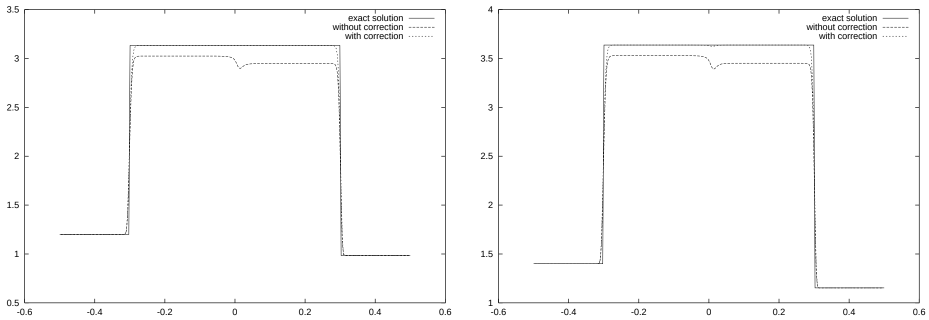

FIG. 2.2 – *Experiment 2 - Pressures 2 and 3*

addition, they all satisfy the same stability estimates but they come with a different cost of numerical evaluation. Here, we have restricted ourselves to the most natural one (along the lines of the first section) but at the expense of the largest numerical cost (we have to solve the nonlinear problem (2.25)). In [19], fully linear correction steps are proposed. In addition, invoking the rotational invariance of the equations under consideration, these methods have been extended to problems with two space variables considering realistic viscosity laws and boundary conditions. We refer the reader to [19].

Chapitre 3

Navier-Stokes equations with several independent pressure laws and explicit predictor-corrector schemes

Ce chapitre fait l'objet d'un article en préparation.

Navier-Stokes equations with several independent pressure laws and explicit predictor-corrector schemes

Chalons C.[1,2] and Coquel F.[3]

Résumé

This work is concerned with the numerical capture of stiff viscous shock solutions of Navier-Stokes equations for complex compressible materials, in the regime of large Reynolds numbers. After [10] and [20], a relevant numerical capture is known to require the satisfaction of an extended set of non classical Rankine-Hugoniot conditions due to the non conservation form of the governing PDE model. Here, we show how to enforce their validity at the discrete level without the need for solving local non linear algebraic problems. Non linearities are bypassed when introducing new averaging techniques which are proved to satisfy all the desirable stability properties when invoking suitable approximate Riemann solutions. A relaxation procedure is proposed to that purpose with the benefit of a fairly simple overall numerical method.

3.1 Introduction

In this paper, we are interested in the numerical approximation of the solutions of the following system in non conservation form :

$$\left\{\begin{array}{l} \partial_t \rho^\epsilon + \partial_x (\rho u)^\epsilon = 0, \\ \partial_t (\rho u)^\epsilon + \partial_x (\rho u^2 + \sum_{i=1}^N p_i)^\epsilon = \epsilon \partial_x (\sum_{i=1}^N \mu_i \partial_x u^\epsilon), \\ \partial_t p_i^\epsilon + \partial_x (p_i u)^\epsilon + (\gamma_i - 1) p_i^\epsilon \partial_x u^\epsilon = \epsilon \mu_i (\partial_x u^\epsilon)^2, \quad i = 1, ..., N, \end{array}\right. \tag{3.1}$$

for a given fixed but small $\epsilon > 0$ under the assumption of a large Reynolds number. This PDE system readily stands as a natural extension of the usual Navier-Stokes equations when a single pressure is involved in the momentum equation. Here, N independent pressure laws occur and are governed by their own PDE, in complete symmetry with the equation governing a single pressure. Then the sum of all these pressures, $p := \sum_{i=1}^N p_i$, enters the momentum equation. Several PDE models from the physics can be identified under this form. Berthon [10] was the first to recognize that the so-called two transport models (namely the $k - \epsilon$ or related models) from the compressible turbulence setting must be actually understood within the present framework for extended Navier-Stokes equations with $N = 2$. Chalons [19] has then shown that more sophisticated turbulent models (the so-called muti-scale models) also naturally fall within the present frame with $N \geq 2$ arbitrary pressures. Let us mention that other models from the physics also enter the proposed setting : namely the

1. O.N.E.R.A., B.P. 72, 29 avenue de la Division Leclerc, 92322 Châtillon Cedex, France.
E-mail: Christophe.Chalons@onera.fr.

2. Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, U.M.R. 7641, École Polytechnique, 91128 Palaiseau Cedex, France.
E-mail: chalons@cmapx.polytechnique.fr

3. Laboratoire JLL & Centre National de la Recherche Scientifique, U.M.R. 7598, Boîte courrier 187, 75252 Paris Cedex 05, France.
E-mail: coquel@ann.jussieu.fr

multi-fluid models introduced by Lagoutière [79] but also other complex gas mixtures analyzed by Truesdell [137].

Here we assume that each of the independent pressures p_i follows a polytropic law with its own adiabatic exponent $\gamma_i > 1,\ i = 1, ..., N$. Next, the viscosity coefficients μ_i entering the system (3.1) denote non negative real numbers but with the requirement of a positive sum, namely $\mu := \sum_{i=1}^{N} \mu_i > 0$.

As a consequence of the thermodynamic closure equations, it can be easily checked (for the details, see [10] with $N = 2$ and [19] with $N > 2$) that the first order underlying system in (3.1) is non linear hyperbolic over the following natural phase space :

$$\Omega = \{\mathbf{V} := (\rho, \rho u, \{p_i\}_{i=1,...,N}) \in I\!R^{N+2} / \rho > 0, \rho u \in I\!R, \{p_i > 0\}_{i=1,...,N}\},$$

from which vacuum is (and will be hereafter) excluded for simplicity in the discussion. Three distinct eigenvalues are in order, namely $u-c < u < u+c$ where the sound speed follows from $c^2 := \sum_{i=1}^{N} \gamma_i p_i/\rho > 0$. The two extreme fields are seen to be genuinely non linear from the assumption $\gamma_i > 1,\ i = 1, ..., N$. By this property, the solutions of the hyperbolic system with viscous perturbations (3.1) are known to generally develop stiff zones of transition often referred in the literature as to viscous shock profiles (see Gilbarg [53] for instance). The numerical approximation of these solutions is of primary importance in the present work. The assumption of a large Reynolds number makes this issue to be particularly challenging since the PDE model (3.1) naturally writes in non conservation form. Actually, Berthon [10] and Chalons [19] have proved that (3.1) cannot recast equivalently for smooth solutions in full conservation form, unless restrictive modelling assumptions are specifically addressed. The following statement proved in [19] gives the existence of solely a non trivial additional conservation law :

Proposition 13 *Smooth solutions of (3.1) satisfy the following conservation law :*

$$\partial_t(\rho E)^\epsilon + \partial_x(\rho H u)^\epsilon = \epsilon \partial_x(\mu u^\epsilon \partial_x u^\epsilon), \tag{3.2}$$

where the total energy ρE and the total enthalpy ρH respectively read :

$$\rho E = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^{N} \frac{p_i}{\gamma_i - 1}, \quad \rho H = \rho E + p.$$

These solutions also obey the following N entropy balance equations :

$$\partial_t s_i^\epsilon + u^\epsilon \partial_x s_i{}^\epsilon = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\epsilon \gamma_i}} (\partial_x u^\epsilon)^2 \geq 0, \quad i = 1, ..., N, \tag{3.3}$$

where the specific entropies are given by $s_i = p_i/\rho^{\gamma_i}$ for $i = 1, ..., N$.

Note from (3.3) that classical non linear transforms in the s_i yield further additional balance equations for governing $\varphi(s_1, ..., s_N)$ where $\varphi : I\!R^N \to I\!R$ denotes any given arbitrary smooth function. But without specific assumptions on the γ_i or the μ_i (see [19] for the details), none of them gives rise to an additional conservation law. The lack of a fully conservative reformulation of (3.1) has been shown in [10] when $N = 2$ and then in [19] for arbitrary $N \geq 2$ to be responsible for the negative issue that classical splitting methods for (3.1) produce approximate solutions which grossly disagree with the expected exact ones. The origin of such a failure can be understood when addressing the equivalent form :

$$\left\{\begin{array}{l}\partial_t\rho^\epsilon + \partial_x(\rho u)^\epsilon = 0,\\ \partial_t(\rho u)^\epsilon + \partial_x(\rho u^2+p)^\epsilon = \epsilon\partial_x(\mu\partial_x u^\epsilon),\\ \partial_t(\rho E)^\epsilon + \partial_x(\rho H u)^\epsilon = \epsilon\partial_x(\mu u^\epsilon\partial_x u^\epsilon),\\ \partial_t s_i^\epsilon + u^\epsilon\partial_x s_i{}^\epsilon = \epsilon\mu_i\frac{\gamma_i-1}{\rho^{\epsilon\gamma_i}}(\partial_x u^\epsilon)^2, \quad i=1,...,N-1,\end{array}\right. \tag{3.4}$$

so that smooth solutions of (3.4) must obey the following additional balance equation :

$$\partial_t s_N^\epsilon + u^\epsilon\partial_x s_N{}^\epsilon = \epsilon\mu_N\frac{\gamma_N-1}{\rho^{\epsilon\gamma_N}}(\partial_x u^\epsilon)^2. \tag{3.5}$$

Here the specific entropy s_N must be obviously understood as a non linear function of the unknown $\mathbf{u} := (\rho, \rho u, \rho E, \{s_i\}_{i=1,...,N-1})$. After Berthon [10] and Chalons [19], it is essential to notice that the $(N-1)$ entropy production rates in (3.4) must be kept in balance with the last one in (3.5) according to :

$$\mu_N\frac{\rho^{\epsilon\gamma_i}}{\gamma_i-1}\{\partial_t s_i^\epsilon + u^\epsilon\partial_x s_i^\epsilon\} = \mu_i\frac{\rho^{\epsilon\gamma_N}}{\gamma_N-1}\{\partial_t s_N(\mathbf{u}^\epsilon) + u^\epsilon\partial_x s_N(\mathbf{u}^\epsilon)\}, \quad i=1,...,N-1. \tag{3.6}$$

The key issue is to notice that these identities are by contrast to (3.3) no longer ϵ-weighted but are instead driven by the ratios of the viscosities μ_i/μ_N. Notice that again, the specific entropy s_N entering (3.6) has to be understood as a non linear function of $\mathbf{u}$, with $\mathbf{u}$ solution of (3.4) so as to satisfy simultaneously the $(N-1)$ balance relations (3.6). Here stays the main difficulty. With this observation, turning considering the numerical issue, one has to evolve the approximate solution $\mathbf{u}_h$ according to (3.4) so that as a byproduct, s_N evaluated on $\mathbf{u}_h$ updates according to (3.6). For mesh scale of practical importance, viscous shock profiles stay under resolved and this lack of resolution implies in turn that the $(N-1)$ entropy production rates in (3.4) are under estimated. For prescribed updates of $(\rho, \rho u, \rho E)$, such under estimations can be seen in turn to imply that the N^{th} production rate in (3.5) is over estimated (see the proofs in Berthon [10], Chalons [19], Chalons and Coquel [20]). The specific entropies $(s_i)_h$ with $i=1,...,N-1$ and the non linear function $s_N(\mathbf{u}_h)$ cannot thus evolve in the balance put forward in (3.6). A failure in satisfying (3.6) has been proved to result in large errors between approximate and exact solutions while the authors in [10], [19] and [20] have established that solving (3.4) when enforcing for validity the $(N-1)$ relations (3.6) yields approximate solutions in fairly good agreement with exact ones. Roughly speaking, the numerical method proposed in these works can be understood as a predictor-corrector strategy which consists first in approximating the solutions of (3.4) and then, in a second step, in correcting the updated unknown $\mathbf{u}_h$ so as to satisfy the following equivalent form of (3.4) :

$$\left\{\begin{array}{l}\partial_t\rho^\epsilon + \partial_x(\rho u)^\epsilon = 0,\\ \partial_t(\rho u)^\epsilon + \partial_x(\rho u^2+p)^\epsilon = \epsilon\partial_x(\mu\partial_x u^\epsilon),\\ \partial_t(\rho E)^\epsilon + \partial_x(\rho H u)^\epsilon = \epsilon\partial_x(\mu u^\epsilon\partial_x u^\epsilon),\\ \mu_N\frac{\rho^{\epsilon\gamma_i}}{\gamma_i-1}\{\partial_t s_i^\epsilon + u^\epsilon\partial_x s_i^\epsilon\} = \mu_i\frac{\rho^{\epsilon\gamma_N}}{\gamma_N-1}\{\partial_t s_N(\mathbf{u}^\epsilon) + u^\epsilon\partial_x s_N(\mathbf{u}^\epsilon)\}, \quad i=1,...,N-1.\end{array}\right. \tag{3.7}$$

Let us underline that the above description of the predictor-corrector strategy is rough. Indeed and to prove the relevance of this procedure, namely the positivity of all the independent pressures together with entropy inequalities, it was necessary to deal with non linear transforms in the s_i so as to obey the following convexity properties:

Lemma 10 *Let be given for $i = 1, ..., N$, N smooth strictly decreasing functions $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

$$\frac{f_i''(x)}{f_i'(x)} < -\frac{\alpha - 1}{\alpha} x, \quad \textit{for all } x > 0, \tag{3.8}$$

where we have set $\alpha = \max_{1 \le i \le N} \gamma_i$. Let us then define the N entropies $\rho S_i = \rho f_i(s_i)$. Then the following convexity properties are in order:

(i) the mapping

$$\{\rho E\}(\rho, \rho u, \{\rho S_i\}_{1 \le i \le N}) = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^{N} \frac{\rho^{\gamma_i}}{\gamma_i - 1} f_i^{-1}(\frac{\rho S_i}{\rho}) \tag{3.9}$$

is strictly convex,

(ii) the mapping in (3.9) is invertible with respect to any given entropy, say S_N, and the mapping

$$\{\rho S_N\}(\rho, \rho u, \rho E, \{\rho S_i\}_{1 \le i \le N-1}) = \rho f_N(\rho E - \frac{(\rho u)^2}{2\rho} - \sum_{i=1}^{N} \frac{\rho^{\gamma_i}}{\gamma_i - 1} f_i^{-1}(\frac{\rho S_i}{\rho})) \tag{3.10}$$

is strictly convex.

The proof of this technical result easily follows from the proof given in [10] with $N = 2$. Observe that choosing a particular family of N functions f_i just amounts to favour a given complete set of equations of state when defining the independent temperatures by:

$$T_i = -\frac{\rho^{\gamma_i - 1}}{\gamma_i - 1}(f_i^{-1})'(\frac{\rho S_i}{\rho}), \quad i = 1, ..., N. \tag{3.11}$$

For a given fixed family, the $(N-1)$ balance equations (3.6) equivalently rewrite

$$\mu_N T_i \{\partial_t(\rho S_i)^\epsilon + \partial_x(\rho S_i u)^\epsilon\} = \mu_i T_N \{\partial_t(\rho S_N)^\epsilon + \partial_x(\rho S_N u)^\epsilon\}, \quad i = 1, ..., N-1,$$

where ρS_N has to be understood as the non linear function given in (3.10). Let us underline that the condition $\alpha = \max_{1 \le i \le N} \gamma_i > 1$ precludes the natural choice $f_i = Id$ for any given $i \in \{1, ..., N\}$. Non linear transforms in the s_i have been thus considered in [10] and [20] for the sake of stability properties but at the expense of the need for solving a non linear algebraic problem at the correction step.

In the present work, we prove that the non linear algebraic problem can be bypassed when choosing $f_i = Id$ for all $i = 1, ..., N$ while still achieving the required stability properties. To circumvent the lack of convexity, we shall introduce suitable averaging procedures which no longer rely on the classical Lebesgue measure involved in the usual Godunov approach. Such averaging procedures will be easily seen to receive fairly simple evaluations when considering classical approximate Riemann solvers for the hyperbolic underlying system in (3.4). We shall in particular derive such a suitable Riemann solver when approximating the solutions of the first order underlying system in (3.4) by those of a relaxation system.

3.2 Classical averagings and stability estimates

The present section aims at briefly describing a variant of the predictor-corrector procedure developed in [20] when revisiting the classical averaging procedures in terms of stability estimates. This variant will turn out to be more convenient for our forthcoming numerical purposes and the discussion we propose below will naturally motivate the need for other suitable averaging techniques. Here, we propose to approximate the smooth solutions of (3.1) when addressing the equivalent formulation :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = \epsilon \partial_x (\mu \partial_x u), \\ \partial_t s_i + u \partial_x s_i = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i}} (\partial_x u)^2, \quad i = 1, ..., N, \end{cases} \tag{3.12}$$

that is to say when the total energy is understood as a non linear function of the unknown $\mathbf{u} := (\rho, \rho u, \{s_i\}_{i=1,...,N})$, so that smooth solutions of (3.12) obey the following non trivial additional equation :

$$\partial_t \rho E + \partial_x \rho H u = \epsilon \partial_x (\mu u \partial_x u).$$

The reason for considering the total energy as a non linear function of the unknown will arise naturally in the next section. Note that for simplicity in the notations, the superscript ϵ has been (and will be hereafter) omitted.
Let be given $\mathbf{u}_h(x, t^n)$ some approximate solution of (3.12) at time t^n. This approximate solution will be classically assumed to be piecewise constant with :

$$\mathbf{u}_h(x, t^n) = \mathbf{u}_j^n, \quad \text{for all} \quad x \in (x_{j-1/2}, x_{j+1/2}), \quad j \in Z,$$

where for simplicity $(x_{j+1/2} - x_{j-1/2}) = \Delta x > 0$. In order to advance it to the next time level t^{n+1}, we first propose two steps based on a splitting strategy between the convective and the diffusive part. Despite this procedure sounds quite natural, we aim at proving that a non linear correction step is needed when invoking classical Riemann solvers for solving the convective part. The relevance of the correction step will only follow when focusing on entropy satisfying Riemann solvers.

Step 1 ($t^n \to t^{n+1=}$)
Solving the underlying first order system in (3.12) clearly amounts to consider weak solutions of the following non linear hyperbolic system :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0, \\ \partial_t s_i + u \partial_x s_i = 0, \quad i = 1, ..., N, \end{cases} \tag{3.13}$$

but under this form, the candidate for defining a Lax entropy pair, namely the total energy, has no natural convexity property. This issue seems to rise a first difficulty. Indeed, weak solutions need to be selected according to some entropy criterion. But other suitable criteria, equivalent (in the present setting) to the requirement of an entropy inequality, are actually at hand to uniquely define Riemann solutions of (3.13) (see indeed Lax [84]). The very difficulty actually arises from the fact that we shall need a control over the discrete energy understood as a function of the approximate solution in order to propose a relevant numerical procedure. The required control will be seen to be directly inferred from a discrete energy like inequality. As it is well-known, such a control heavily relies on the convexity property of the total energy with respect to the (discrete) unknown. This naturally motivates us to solve instead of (3.13) the following Cauchy problem with initial data $\overline{\mathbf{w}}_h(x,t^n) := (\rho, \rho u, \{\rho f_i(s_i)\}_{i=1,...,N})_h(x,t^n)$:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0, \\ \partial_t \rho S_i + \partial_x \rho S_i u = 0, \quad i = 1, ..., N, \end{cases} \tag{3.14}$$

where we have considered a fixed family of N non linear functions f_i given under the requirement (3.8). As a consequence, the total energy ρE is convex in the new unknown $\overline{\mathbf{w}}$, and the pair $(\rho E, \rho H u)$ can now serve as a Lax entropy pair for selecting the weak solutions of (3.14) according to :

$$\partial_t \rho E(\overline{\mathbf{w}}) + \partial_x \rho H u(\overline{\mathbf{w}}) \leq 0. \tag{3.15}$$

It is worthy to noticing that entropy weak solutions of (3.14) actually obey in the usual sense of the distributions the following transport equations for each of the specific entropies :

$$\partial_t s_i + u \partial_x s_i = 0, \quad \mathcal{D}'. \tag{3.16}$$

Indeed, it can be easily checked from the Rankine-Hugoniot conditions associated with (3.14) that the solutions u and S_i (so that s_i) cannot be discontinuous simultaneously. So that the non conservative products entering (3.16) are never ambiguous. The transport equations property, valid for all $i = 1, ..., N$, is actually the main reason for rejecting the total energy as a non linear function of the unknown and will be used intensively in the next section to introduce new averaging techniques. In order to solve the Cauchy problem (3.14)-(3.15) for small times, we make use of the celebrated Godunov method so as to consider a sequence of non interacting exact Riemann solutions under the CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{\overline{\mathbf{w}}}(|u-c|(\overline{\mathbf{w}}), |u+c|(\overline{\mathbf{w}})) \leq \frac{1}{2}, \tag{3.17}$$

for all the $\overline{\mathbf{w}}$ under consideration. For the times under consideration, the solution is then obtained as the superposition of these Riemann solutions. With this solution, let us now invoke the classical averagings in order to produce a piecewise constant update :

$$\overline{\mathbf{w}}_j^{n+1=} := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \overline{\mathbf{w}}_h(x, t^n + \Delta t) dx, \quad j \in Z. \tag{3.18}$$

Note that by the monotonicity assumptions on each of the f_i, the piecewise approximate solution $\mathbf{u}_h(x, t^{n+1=})$ is uniquely recovered from (3.18). But from the non linearity in the f_i, observe that the resulting update of each of the s_i is actually such that :

$$(s_i)_j^{n+1=} := \{s_i\}(\overline{\mathbf{w}}_j^{n+1=}) \neq \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{s_i\}(\overline{\mathbf{w}}_h(x, t^n + \Delta t)) dx =$$

$$(s_i)_j^n - \frac{\Delta t}{\Delta x}(u^\star)^{j+1/2,-}((s_i)_{j+1}^n - (s_i)_j^n) - \frac{\Delta t}{\Delta x}(u^\star)^{j-1/2,+}((s_i)_j^n - (s_i)_{j-1}^n), \quad j \in Z, \tag{3.19}$$

where $(u^\star)^{j+1/2}$ denotes the speed of the contact discontinuity in the Riemann problem at the interface $x_{j+1/2}$. The identity in the right-hand side of (3.19) easily results from the transport equations (3.16). As a consequence, we infer that in general:

$$(s_i)_j^{n+1=} - (s_i)_j^n + \frac{\Delta t}{\Delta x}(u^\star)_-^{j+1/2}((s_i)_{j+1}^n - (s_i)_j^n) + \frac{\Delta t}{\Delta x}(u^\star)_+^{j-1/2}((s_i)_j^n - (s_i)_{j-1}^n) \neq 0, \tag{3.20}$$

and it is not known if $(s_i)_j^{n+1=}$ solves a discrete version of (3.16). Here the benefit of averaging each of the ρS_i instead of the s_i is that the following in-cell total energy inequality holds under the CFL restriction (3.17):

$$\{\rho E\}(\overline{\mathbf{w}}_j^{n+1=}) \leq (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n, \quad j \in Z, \tag{3.21}$$

by the well-known Jensen inequality applied to the convex mapping $\{\rho E\}(\overline{\mathbf{w}})$, together with the following discrete conservation laws:

$$(\rho S_i)_j^{n+1=} = (\rho S_i)_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho S_i u\}_{j+1/2}^n, \quad j \in Z, \tag{3.22}$$

for some quite classical definitions of the numerical flux functions.

Step 2 $(t^{n+1=} \to t^{n+1-})$
Let us consider $\overline{\mathbf{w}}_h(x, t^{n+1=})$ the solution of (3.14)-(3.15) at time Δt. This one now naturally serves as initial data for solving the viscous perturbations for times $t \in [0, \Delta t]$:

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t \rho u = \epsilon \partial_x(\mu \partial_x u), \\ \partial_t \rho S_i = -\epsilon \frac{\mu_i}{T_i}(\partial_x u)^2, \quad i = 1, ..., N, \end{cases} \tag{3.23}$$

for the temperatures T_i defined in (3.11), that is to say since the momentum equation is decoupled from the entropies:

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t \rho u = \epsilon \partial_x(\mu \partial_x u), \\ \partial_t s_i = \epsilon \mu_i \frac{\gamma_i - 1}{(\rho)^{\gamma_i}}(\partial_x u)^2, \quad i = 1, ..., N. \end{cases} \tag{3.24}$$

The numerical approximation of the solution of (3.23), or equivalently of (3.24), has received a particular attention in [20]. Extending the work by Berthon in [10], these authors have proved that convenient finite differences formulae can be proposed so that the total energy, understood as a function of the updated unknown, evolves in time consistently with:

$$\partial_t \rho E = \epsilon \partial_x(\mu u \partial_x u), \tag{3.25}$$

while the N specific entropies obey by construction a discrete version of:

$$\mu_N \frac{\rho^{\gamma_i}}{\gamma_i - 1}\partial_t s_i = \mu_i \frac{\rho^{\gamma_N}}{\gamma_N - 1}\partial_t s_N, \quad i = 1, ..., N-1. \tag{3.26}$$

More precisely, the solution of (3.24) at time Δt is approximated by:

$$\begin{cases} \rho_j^{n+1-} = \rho_j^{n+1=}, \\ (\rho u)_j^{n+1-} = (\rho u)_j^{n+1=} + \Delta t \overline{\partial_x(\mu \partial_x u)}_j^{n+1-}, \\ (s_i)_j^{n+1-} = (s_i)_j^{n+1=} + \Delta t \mu_i \frac{\gamma_i - 1}{(\rho_j^{n+1-})^{\gamma_i}} \overline{(\partial_x u)^2}_j^{n+1-}, \quad i = 1, ..., N, \end{cases} \tag{3.27}$$

where the precise definition of the discrete operator $\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}$, given in [20] (see also [10]), results from a conservative implicit in time method for solving u. Once this implicit scheme is solved, the discrete operator $\overline{(\partial_x u)^2}_j^{n+1-}$ can receive an explicit definition so that in one hand $\overline{(\partial_x u)^2}_j^{n+1-} \geq 0$, while on the second hand:

$$\{\rho E\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, \{(s_i)_j^{n+1-}\}_{i=1,\dots,N}) = (\rho E)_j^{n+1=} + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \quad (3.28)$$

where the resulting discrete quantity $\overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}$ is a consistent and conservative discretization of the expected exact operator (see [20] for the details). Let us emphasize that the N last updating formulae in (3.27) easily imply for all $i = 1, \dots, N-1$:

$$\mu_N \frac{(\rho_j^{n+1-})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1-} - (s_i)_j^{n+1=}\} = \mu_i \frac{(\rho_j^{n+1-})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1-} - (s_N)_j^{n+1=}\}, \quad (3.29)$$

which is nothing but a consistent discrete form of (3.26). To conclude this brief presentation, it is worthy to noticing that on the ground of the discrete operators we have just introduced, we immediately get from the updating formulae (3.27) equivalent discrete formulae for approximating the system (3.23):

$$(\rho S_i)_j^{n+1-} = (\rho S_i)_j^{n+1=} - \Delta t \frac{\mu_i}{\overline{T}_{ij}^{n+1-}} \overline{(\partial_x u)^2}_j^{n+1-}, \quad i = 1, \dots N,$$

where

$$\overline{T}_{ij}^{n+1-} = \frac{(\rho_j^{n+1-})^{\gamma_i - 1}}{\gamma_i - 1} \begin{cases} (f_i^{-1})'(f_i((s_i)_j^{n+1=})) \text{ if } f_i((s_i)_j^{n+1-}) = f_i((s_i)_j^{n+1=}), \\ \frac{f_i^{-1} o f_i((s_i)_j^{n+1-}) - f_i^{-1} o f_i((s_i)_j^{n+1=})}{f_i((s_i)_j^{n+1-}) - f_i((s_i)_j^{n+1=})} \text{ otherwise.} \end{cases}$$

As a consequence of the equivalence, we have:

$$\{\rho E\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, \{(\rho S_i)_j^{n+1-}\}_{i=1,\dots,N}) = (\rho E)_j^{n+1=} + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \quad (3.30)$$

while on the same time, for $i = 1, \dots, N-1$:

$$\mu_N \overline{T}_{ij}^{n+1-}\{(\rho S_i)_j^{n+1-} - (\rho S_i)_j^{n+1=}\} = \mu_i \overline{T}_{Nj}^{n+1-}\{(\rho S_N)_j^{n+1-} - (\rho S_N)_j^{n+1=}\}. \quad (3.31)$$

These remarks will be useful below.

Step 3 $(t^{n+1-} \to t^{n+1})$ *: Correction procedure*

As they stand, the first two steps we have just described provide us with a formally consistent approximation of the solutions of (3.1). But these two steps cannot result in a relevant numerical method since by construction, the L^1 norm of the total energy is (strictly) decreasing in time by virtue of the next inequality:

$$\{\rho E\}(\overline{\mathbf{w}}_j^{n+1-}) \leq (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \quad (3.32)$$

as a direct consequence of the inequality (3.20) and the equality (3.28). A correction procedure for restoring the conservation of the total energy at the discret level is therefore needed. Despite that inequality (3.32) is responsible for this need, the same inequality (3.32) actually authorizes the correction step to take place, that is to say when preserving the positivity of each of the entropies. Our main motivation here, after [10] and [20], is to propose a correction step so that the final updated unknown satisfies the $(N-1)$ identities (3.31) at the discrete level while preserving the conservation of the total energy. In that aim, we are led to keep unchanged the definitions of the density and the momentum:

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad j \in Z,$$

while redefining the N entropies so that in the one hand:

$$(\rho E)_j^{n+1} = (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-},$$

while on the second hand we arrive at a consistent discretization of:

$$\mu_N \frac{\rho^{\gamma_i}}{\gamma_i - 1}\{\partial_t s_i + u\partial_x s_i\} = \mu_i \frac{\rho^{\gamma_N}}{\gamma_N - 1}\{\partial_t s_N(\mathbf{u}) + u\partial_x s_N(\mathbf{u})\}, \quad i = 1, ..., N-1.$$

In spite of the validity of the $(N-1)$ relations (3.29):

$$\mu_N \frac{(\rho_j^{n+1-})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1-} - (s_i)_j^{n+1=}\} = \mu_i \frac{(\rho_j^{n+1-})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1-} - (s_N)_j^{n+1=}\},$$

the inequalities (3.20):

$$(s_i)_j^{n+1=} \neq (s_i)_j^n - \frac{\Delta t}{\Delta x}(u^\star)^{j+1/2,-}((s_i)_{j+1}^n - (s_i)_j^n) - \frac{\Delta t}{\Delta x}(u^\star)^{j-1/2,+}((s_i)_j^n - (s_i)_{j-1}^n),$$

preclude the specific entropies s_i from being suitable candidates. We are thus necessarily left with the non linear form of the entropies S_i which by definition obey similar balance relations:

$$\mu_N T_i\{\partial_t \rho S_i + \partial_x \rho S_i u\} = \mu_i T_N\{\partial_t \rho S_N + \partial_x \rho S_N u\}, \quad i = 1, ..., N-1. \tag{3.33}$$

Note that both set of equalities (3.31) and (3.22) now make possible the derivation of a correction step consistent with (3.33). Indeed, it suffices to redefine $(\rho S_i)_j^{n+1}$ as the solution of the following $(N-1)$ equations:

$$\mu_N \overline{T_i}_j^{n+1-}\{(\rho S_i)_j^{n+1} - (\rho S_i)_j^{n+1=}\} = \mu_i \overline{T_N}_j^{n+1-}\{(\rho S_N)_j^{n+1} - (\rho S_N)_j^{n+1=}\},$$

with $i = 1, ..., N-1$, supplemented with the following non linear equation:

$$\{\rho E\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, \{(\rho S_i)_j^{n+1}\}_{i=1,...,N}) = (\rho E)_j^{n+1}. \tag{3.34}$$

Invoking the energy inequality $(\rho E)_j^{n+1-} \leq (\rho E)_j^{n+1}$, a proof similar to the one proposed in [20] gives the existence of a unique solution with the property that

$$0 < (\rho S_i)_j^{n+1} \leq (\rho S_i)_j^{n+1=}, \quad i = 1, ..., N. \tag{3.35}$$

In practice, an iterative procedure is required to solve (3.34). As a consequence of these estimates, each of the partial pressures stays positive as soon as $\rho_j^{n+1} > 0$ while the following entropy inequalities are met (see (3.22)):

$$(\rho S_i)_j^{n+1} - (\rho S_i)_j^n + \frac{\Delta t}{\Delta x}\Delta\{\rho S_i u\}_{j+1/2}^n \leq 0, \quad i = 1, ..., N.$$

To conclude this section, let us again underline that the crucial stability estimates (3.35) only hold true because of the energy inequality (3.32). The validity of this inequality relies on using classical averagings for non linear versions of the specific entropies s_i. We show in the next section how to suitably average directly the s_i so as to recover the required energy inequality. The benefit will be a fully explicit correction step by opposition to (3.34) while still achieving the same stability properties.

3.3 A new averaging procedure

The predictor-corrector method we now introduce comes along the main lines of the three steps we have described in the previous section. Essentially, the first step devoted to the approximation of the underlying non linear hyperbolic system will receive a suitable modification so as to end up with a fully explicit but still relevant correction step. The second step will remain unchanged.
Here assuming again a piecewise constant approximate solution $\mathbf{u}_h(x,t^n)$ to be given at the date t^n, this one is advanced to the next intermediate time level $t^{n+1=}$ when solving the Cauchy problem (3.14) in the $\overline{\mathbf{w}} := (\rho, \rho u, \{\rho S_i\}_{i=1,\dots,N})$ variable. So that again the total energy plays the role of a Lax entropy for selecting the weak solutions under consideration. The solution $\overline{\mathbf{w}}(x,t)$ of this problem is obtained under the CFL condition (3.17) as the superposition of non interacting Riemann solutions for times t less than Δt. Here instead of projecting onto the constants the ρS_i at time $t^{n+1=} = t^n + \Delta t$, we suggest to consider suitable averagings of $s_i(\overline{\mathbf{w}}(x,t^{n+1=}))$. Note that the classical projections onto the constants:

$$(s_i)_j^{n+1=} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} s_i(\overline{\mathbf{w}}(x,t^{n+1=}))dx, \quad i = 1,\dots,N,$$

cannot provide us with a relevant definition to make the total energy to decrease simply because of the lack of convexity of the mapping $\mathbf{u} \to \{\rho E\}(\mathbf{u})$. To bypass this lack of convexity, we instead suggest to consider the following averaging procedures.

Definition 4 *Under the CFL condition (3.17), the solution $\overline{\mathbf{w}}$ of the Cauchy problem (3.14) is given at time $t^{n+1=}$ the following averaging formulae over each cell $(x_{j-1/2}, x_{j+1/2})$, $j \in Z$:*

$$\begin{aligned}\rho_j^{n+1=} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx,\\ (\rho u)_j^{n+1=} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho u\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx,\end{aligned} \tag{3.36}$$

while the specific entropies are projected onto the constants according to:

$$(s_i)_j^{n+1=} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{s_i\}(\overline{\mathbf{w}}(x,t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}_j^{n+1=}}(\overline{\mathbf{w}}(x,t^{n+1=}))dx, \quad j \in Z, \tag{3.37}$$

where we have set:

$$\overline{\rho^{\gamma_i}}_j^{n+1=} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx.$$

In other words, both the density and the momentum receive classical averaging formulae based on the Lebesgue measure, while by contrast the specific entropies are averaged when invoking probability-like measures: *i.e.* positive Borel measures with unit total mass. Consequently, the positivity of each of the specific entropies is preserved by construction. A well-known consequence of the classical averagings (3.36) is that both density and momentum can be given an equivalent updating formula in full conservation form:

$$\begin{aligned}\rho_j^{n+1=} &= \rho_j^n - \tfrac{\Delta t}{\Delta x}\Delta\{\rho u\}_{j+1/2}^n,\\ (\rho u)_j^{n+1=} &= (\rho u)_j^n - \tfrac{\Delta t}{\Delta x}\Delta\{\rho u^2+p\}_{j+1/2}^n,\end{aligned}\tag{3.38}$$

where classically $\{\rho u\}_{j+1/2}^n = \{\rho u\}(\overline{\mathbf{w}}(0^+,\mathbf{u}_j^n,\mathbf{u}_{j+1}^n))$ with a similar definition for $\{\rho u^2+p\}_{j+1/2}^n$. By contrast, the original projection formulae (3.37) cannot give rise to a discrete conservation form for governing the specific entropies since these variables already obey at the PDE level the following equation:

$$\partial_t s_i + u\partial_x s_i = 0, \quad \mathcal{D}', \quad i=1,...,N. \tag{3.39}$$

We shall prove later on that the averaging formulae (3.37) actually yield a discrete version of (3.39) under the form:

$$(s_i)_j^{n+1=} = (s_i)_j^n - \frac{\Delta t}{\Delta x}v_i^{j-1/2,+}\{(s_i)_j^n-(s_i)_{j-1}^n\} - \frac{\Delta t}{\Delta x}v_i^{j+1/2,-}\{(s_i)_{j+1}^n-(s_i)_j^n\}, \tag{3.40}$$

where the discrete velocity $v_i^{j+1/2*}$ will be given a consistent definition.
The unusual projection formulae (3.37) will be given further insight hereafter but they are primarily dictated by the following result:

Lemma 11 *Let the approximate solution be advanced to the next time level $t^{n+1=}$ according to (3.36)-(3.37). Then, under the CFL condition (3.17), the following in-cell energy inequalities are satisfied for all $j\in Z$:*

$$\{\rho E\}(\mathbf{u}_j^{n+1=}) - (\rho E)_j^n + \lambda\Delta\{\rho\mathcal{H}u\}^+(\mathbf{u}_j^n,\mathbf{u}_{j+1}^n) \le 0, \tag{3.41}$$

where classically:

$$\{\rho\mathcal{H}u\}^+(\mathbf{u}_j^n,\mathbf{u}_{j+1}^n) = \{(\rho E+p)u\}(\overline{\mathbf{w}}(0^+,\mathbf{u}_j^n,\mathbf{u}_{j+1}^n)). \tag{3.42}$$

Proof Classical considerations (see [57]) imply from the Lax entropy inequality (3.15) the following in-cell inequality:

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}\{\rho E\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx \le (\rho E)_j^n - \lambda\Delta\{\rho\mathcal{H}u\}^+(\mathbf{u}_j^n,\mathbf{u}_{j+1}^n), \tag{3.43}$$

where the numerical energy flux is given by (3.42). Let us now notice that the definitions of $(s_i)_j^{n+1=}$ and $\overline{\rho^{\gamma_i}}_j^{n+1=}$ for all $i=1,...,N$ lead to

$$\begin{gathered}\int_{x_{j-1/2}}^{x_{j+1/2}}\{\rho E\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx = \\ \int_{x_{j-1/2}}^{x_{j+1/2}}\{\frac{(\rho u)^2}{2\rho}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx + \sum_{i=1}^N\int_{x_{j-1/2}}^{x_{j+1/2}}\frac{\{p_i\}}{\gamma_i-1}(\overline{\mathbf{w}}(x,t^{n+1=}))dx = \\ \int_{x_{j-1/2}}^{x_{j+1/2}}\{\frac{(\rho u)^2}{2\rho}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx + \sum_{i=1}^N\frac{(s_i)_j^{n+1=}}{\gamma_i-1}\times\int_{x_{j-1/2}}^{x_{j+1/2}}\{\rho^{\gamma_i}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx.\end{gathered}\tag{3.44}$$

In the one hand, the convexity of the mapping $(\rho, \rho u) \to (\rho u)^2/2\rho$ ensures from the well-known Jensen's inequality :

$$\frac{((\rho u)_j^{n+1=})^2}{2\rho_j^{n+1=}} \le \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\frac{(\rho u)^2}{2\rho}\}(\overline{\mathbf{w}}(x, t^{n+1=}))dx. \tag{3.45}$$

In the second hand, the convexity of the mapping $\rho \to \rho^{\gamma_i}$ ($\gamma_i > 1$) gives again because of the Jensen's inequality :

$$(\rho_j^{n+1=})^{\gamma_i} \le \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\overline{\mathbf{w}}(x, t^{n+1=}))dx, \quad i = 1, ..., N.$$

Since the averaging formulae (3.37) preserve the positivity of the specific entropies, we deduce the required estimate :

$$\begin{aligned} \{\rho E\}(\mathbf{u}_j^{n+1=}) \quad &:= \quad \frac{((\rho u)_j^{n+1=})^2}{2\rho_j^{n+1=}} + \sum_{i=1}^{N} \frac{(\rho_j^{n+1=})^{\gamma_i}}{\gamma_i - 1}(s_i)_j^{n+1=} \\ &\le \quad \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(\overline{\mathbf{w}}(x, t^{n+1=}))dx. \end{aligned} \tag{3.46}$$

This completes the proof.

The energy inequality (3.41) thus makes relevant the unusual averagings (3.36)-(3.37) to define a discrete method for approximating the weak solutions of the non linear hyperbolic system (3.14) in the first step of our splitting strategy. We are now in a position to address the second step devoted to the viscous perturbations. The second step described in the previous section is kept unchanged. Using the same notations, we thus deduce the following statement :

Corollary 3 *The first two steps in the splitting method yield the following updating formulae :*

$$\begin{cases} \rho_j^{n+1-} = \rho_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u\}_{j+1/2}^n, \\ (\rho u)_j^{n+1-} = (\rho u)_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u^2 + p\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}, \\ (s_i)_j^{n+1-} = (s_i)_j^{n+1=} + \Delta t\frac{\gamma_i - 1}{(\rho_j^{n+1-})^{\gamma_i}}\overline{(\partial_x u)^2}_j^{n+1-}, \quad i = 1, ..., N, \end{cases} \tag{3.47}$$

where

$$\overline{(\partial_x u)^2}_j^{n+1-} \ge 0, \quad \text{and} \quad \overline{\partial_x(\mu\partial_x u)}_j^{n+1-}, \tag{3.48}$$

are given consistent finite differences formulae in [20]. Under the CFL condition (3.17), the next energy-like inequality holds on each cell $(x_{j-1/2}, x_{j+1/2})$:

$$\{\rho E\}(\mathbf{u}_j^{n+1-}) \le (\rho E)_j^n - \lambda\Delta\{\rho\mathcal{H}u\}^+(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-}, \tag{3.49}$$

where the last term in (3.49) receives by construction a discrete formula in conservation form.

Let us indeed recall that the discrete formulae (3.47) are such that by construction, the total energy obeys with equality the following identity in conservation form (see [20] and [10] for the details) :

$$\{\rho E\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, \{(s_i)_j^{n+1-}\}_{i=1,...,N}) = (\rho E)_j^{n+1=} + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-}.$$

With this respect, the inequality (3.49) just follows from inequality (3.41).
The proof of Lemma 11 clearly indicates that the energy inequality (3.41) holds strictly generally speaking so that also (3.49). The formulae (3.36)-(3.37)-(3.47) thus make the L^1 norm of the total energy to decrease (strictly) in time. Here again, a correction step to enforce the conservation of the total energy is required. Following the guidelines of the rough description of the method, we are led to keep unchanged the updating formulae for the density and the momentum :

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad \text{for all } j \in Z, \tag{3.50}$$

while defining the N specific entropies $(s_i)_j^{n+1}$ as the solutions of the following $(N-1)$ equations :

$$\mu_N \frac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1} - (s_i)_j^{n+1=}\} = \mu_i \frac{(\rho_j^{n+1})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1} - (s_N)_j^{n+1=}\}, \tag{3.51}$$

supplemented with

$$\sum_{i=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}(s_i)_j^{n+1} = (\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}}. \tag{3.52}$$

Here the total energy $(\rho E)_j^{n+1}$ is defined by

$$(\rho E)_j^{n+1} := (\rho E)_j^n - \lambda\Delta\{\rho\mathcal{H}u\}^+(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \tag{3.53}$$

so as to restore its conservation property at the discrete level. Let us underline that the algebraic problem (3.51)-(3.52) is linear so that the next easy result, which proof is left to the reader, holds :

Proposition 14 *The linear system (3.51) in the unknown $\{(s_i)_j^{n+1}\}_{1\le i\le N}$ admits an unique solution explicitly given by :*

$$\begin{gathered}(s_i)_j^{n+1} = (s_i)_j^{n+1=} + \\ \frac{\mu_i(\gamma_i - 1)}{(\rho_j^{n+1})^{\gamma_i}\sum_{l=1}^N \mu_l}\left((\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} - \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1}(s_l)_j^{n+1=}\right),\end{gathered} \tag{3.54}$$

for all $i = 1, \ldots, N$.

The correction procedure is thus fully explicit by contrast to the technique developed in the previous section. But here again, as expected, the discrete energy inequality (3.49) is essentially responsible for proving the relevance of the present correction step :

Theorem 6 *Under the CFL condition (3.17), the solution (3.54) satisfies the following maximum principles :*

$$(s_i)_j^{n+1} \geq (s_i)_j^{n+1=} \geq \min((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad j \in Z, \quad i = 1, ..., N. \tag{3.55}$$

As a consequence, the numerical method (3.36)-(3.37)-(3.47)-(3.50)-(3.53)-(3.54) preserves the positivity of the specific entropies and hence the positivity of the partial pressures :

$$(p_i)_j^{n+1} := (\rho_j^{n+1})^{\gamma_i}(s_i)_j^{n+1} \geq 0, \quad i = 1, ..., N.$$

Proof Note first that

$$\frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} + \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1}(s_l)_j^{n+1=} \leq \{\rho E\}(\mathbf{u}_j^{n+1-}),$$

since we have $(s_i)_j^{n+1=} \leq (s_i)_j^{n+1-}$ by the positivity property of $\overline{(\partial_x u)^2}_j^{n+1-}$ reported in the Corollary 3. As a consequence, the energy inequality (3.49) valid under the CFL condition (3.17) just recasts as:

$$\frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} + \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1}(s_l)_j^{n+1=} \leq \{\rho E\}(\mathbf{u}_j^{n+1-}) \leq (\rho E)_j^{n+1},$$

by definition of $(\rho E)_j^{n+1}$. Invoking the explicit formula (3.54) for defining each of the specific entropies $(s_i)_j^{n+1}$, the positivity of the viscosity coefficients easily ensures that

$$(s_i)_j^{n+1} \geq (s_i)_j^{n+1=}.$$

The proof will be completed under some suitable CFL condition when proving for validity the convex decomposition (3.40). This will be the matter of the Proposition 16 given hereafter.
The following statement summarizes the main properties of the prediction-correction method we have just proposed.

Corollary 4 *The method (3.36)-(3.37)-(3.47)-(3.50)-(3.53)-(3.54) provides an approximate discrete solution which satisfies the following identities:*

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u\}_{j+1/2}^n, \\ (\rho u)_j^{n+1} = (\rho u)_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u^2 + p\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}, \\ (\rho E)_j^{n+1} = (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-}, \\ \mu_N \dfrac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1} - (s_i)_j^{n+1=}\} = \mu_i \dfrac{(\rho_j^{n+1})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1} - (s_N)_j^{n+1=}\}, \end{cases} \tag{3.56}$$

where with $i = 1, ..., N$:

$$(s_i)_j^{n+1=} := (s_i)_j^n - \frac{\Delta t}{\Delta x}v_i^{j-1/2,+}\{(s_i)_j^n - (s_i)_{j-1}^n\} - \frac{\Delta t}{\Delta x}v_i^{j+1/2,-}\{(s_i)_{j+1}^n - (s_i)_j^n\}, \tag{3.57}$$

for some consistent averaged values $v_i^{j+1/2}$ of the velocity u defined in (3.67) below. In addition, the following N entropy-like inequalities are valid under the CFL condition (3.17):

$$(s_i)_j^{n+1} - (s_i)_j^n + \frac{\Delta t}{\Delta x}v_i^{j-1/2,+}\{(s_i)_j^n - (s_i)_{j-1}^n\} + \frac{\Delta t}{\Delta x}v_i^{j+1/2,-}\{(s_i)_{j+1}^n - (s_i)_j^n\} \geq 0. \tag{3.58}$$

In other words, the proposed method yields a consistent discretization of the equivalent system (3.7) based on the $(N-1)$ generalized jump conditions (3.6). Let us emphasize that inequalities (3.58) are just a discrete version of the following entropy inequalities:

$$\partial_t s_i + u\partial_x s_i = \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i}}(\partial_x u)^2 \geq 0, \quad i = 1, ..., N.$$

We mention that discrete version of the inequalities in the ρS_i are not known at the present moment. But let us stress that the satisfaction of the maximum principles (3.55) on each of the specific entropies s_i are known after Tadmor [135] (see also Khobalatte and Perthame [78]) to hold as a consequence of an infinite number of classical entropy inequalities.
Let us now turn defining the discrete velocities entering the identities (3.57). In that aim, it is convenient to adopt an Harten, Lax, Van Leer type of analysis (see [61]) when introducing the following half averagings at a given interface $x_{j-1/2}$ under the CFL (3.17):

$$(s_i)^{n+1=}_{j+1/2,L} = \frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} s_i(\overline{\mathbf{w}}(x,t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}}(\overline{\mathbf{w}}(x,t^{n+1=}))dx \quad j\in Z, \tag{3.59}$$

where we have set:

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} = \frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx.$$

Symmetrically we define:

$$(s_i)^{n+1=}_{j+1/2,R} = \frac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}} s_i(\overline{\mathbf{w}}(x,t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R}}(\overline{\mathbf{w}}(x,t^{n+1=}))dx \quad j\in Z, \tag{3.60}$$

where:

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} = \frac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}} \{\rho^{\gamma_i}\}(\overline{\mathbf{w}}(x,t^{n+1=}))dx.$$

Notice that the averaging procedures (3.37) for defining each of the $(s_i)^{n+1=}_j$ equivalently rewrite in terms of the above half averages as follows:

$$(s_i)^{n+1=}_j = \frac{\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} \times (s_i)^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} \times (s_i)^{n+1=}_{j+1/2,L}}{\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}}, \tag{3.61}$$

since by construction

$$\overline{\rho^{\gamma_i}}^{n+1=}_j = \frac{1}{2}(\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}). \tag{3.62}$$

Equipped with these notations, let us state :
Proposition 15 *Under the CFL condition (3.17), the half averages (3.59) and (3.60) respectively read:*

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} \times (s_i)^{n+1=}_{j+1/2,L} = \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} \times (s_i)^n_j - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,-}((s_i)^n_{j+1} - (s_i)^n_j),$$

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^{n+1=}_{j+1/2,R} = \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^n_{j+1} - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,+}((s_i)^n_{j+1} - (s_i)^n_j), \tag{3.63}$$

where the discrete quantities $w_i^{j+1/2}$ are given by

$$w_i^{j+1/2,\pm} = w_i^\pm(\mathbf{u}^n_j, \mathbf{u}^n_{j+1}) = \int_0^{(u^\star_{j+1/2})^\pm} \rho^{\gamma_i}(\xi)d\xi, \tag{3.64}$$

with

$$(u^\star_{j+1/2})^+ = \max(u^\star_{j+1/2}, 0), \quad (u^\star_{j+1/2})^- = \min(u^\star_{j+1/2}, 0).$$

Here $\xi \to \rho(\xi)$ denotes the density function in the self-similar exact Riemann solution at the interface $x_{j+1/2}$ while $u^\star_{j+1/2}$ is the speed of propagation of the contact discontinuity. By contruction, the mappings $(\mathbf{u}_L, \mathbf{u}_R) \in \Omega^2 \to w_i^\pm(\mathbf{u}_L, \mathbf{u}_R) \in I\!R$ are locally Lipschitz-continuous with the property that

$$w_i^\pm(\mathbf{u}, \mathbf{u}) = \rho^{\gamma_i} u^\pm \quad \textit{for all} \quad \mathbf{u} \in \Omega.$$

In the definition (3.64), it is tacitly assumed that the Riemann solution under consideration is made of three distinct waves so that the velocity $u^\star$ of the intermediate contact wave is clearly well-defined. In order to extend its definition to the case of a Riemann solution made of a single wave (either a 1-wave or a 3-wave with classical definitions), we propose the following natural convention. When $\mathbf{u}_j$ and $\mathbf{u}_{j+1}$ are connected by a 1-wave, we set $u^\star_{j+1/2} = u^n_{j+1}$ while when connected by a 3-wave, we choose $u^\star_{j+1/2} = u^n_j$. This convention ensures the reported Lipschitz-continuity.

Proof Under the restriction CFL (3.17), the solution $\overline{\mathbf{w}}(x,t)$ coincides for all $(x,t) \in (x_j, x_{j+1}) \times (t^n, t^{n+1=})$ with the exact Riemann solution $\overline{\mathbf{w}}((x - x_{j+1/2})/(t - t^n), \mathbf{u}^n_j, \mathbf{u}^n_{j+1})$. Up to some shift in space and time, each of the specific entropies is easily seen to be given as the following step function propagating with speed $u^\star = u^\star_{j+1/2}$, namely the speed of the contact discontinuity :

$$s_i(x,t) = (s_i)^n_j + ((s_i)^n_{j+1} - (s_i)^n_j)\mathbf{H}(x - u^\star t), \tag{3.65}$$

where $\mathbf{H}$ denotes the usual Heavyside function. Let us restrict ourselves to a positive value $u^\star$. The case of a non positive value will follow from straightforward adaptations. Under this assumption, we clearly get when evaluating the formula definition (3.59) :

$$(s_i)^{n+1=}_{j+1/2,L} = (s_i)^n_j$$

while

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^{n+1=}_{j+1/2,R} =$$
$$\frac{2}{\Delta x}\int_0^{\Delta x/2} \rho^{\gamma_i}(\frac{x}{\Delta t}) \times \{(s_i)^n_j + ((s_i)^n_{j+1} - (s_i)^n_j)\mathbf{H}(x - u^\star \Delta t)\} dx =$$
$$\frac{2\Delta t}{\Delta x}\{(s_i)^n_j \int_0^{u^\star} \rho^{\gamma_i}(\xi) d\xi + (s_i)^n_{j+1} \int_{u^\star}^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi) d\xi\}.$$

We thus easily deduce that

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^{n+1=}_{j+1/2,R} =$$
$$(s_i)^n_{j+1} \times \frac{2\Delta t}{\Delta x}\int_0^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi) d\xi - \frac{2\Delta t}{\Delta x}\int_0^{u^\star} \rho^{\gamma_i}(\xi) d\xi \times ((s_i)^n_{j+1} - (s_i)^n_j),$$

where

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} = \frac{2\Delta t}{\Delta x}\int_0^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi) d\xi.$$

The conclusion easily follows.
We are now in position to establish :

Proposition 16 *Under the CFL condition (3.17), the updating formulae (3.37) read:*

$$(s_i)_j^{n+1=} = (s_i)_j^n - \frac{\Delta t}{\Delta x} v_i^{j-1/2,+}((s_i)_j^n - (s_i)_{j-1}^n) - \frac{\Delta t}{\Delta x} v_i^{j+1/2,-}((s_i)_{j+1}^n - (s_i)_j^n), \quad (3.66)$$

where we have set:

$$v_i^{j+1/2,\pm} = v_i^{\pm}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = \int_0^{(u_{j+1/2}^\star)^\pm} \rho^{\gamma_i}(\xi) d\xi / \overline{\rho^{\gamma_i}}_j^{n+1=}. \quad (3.67)$$

By contruction, the mappings $(\mathbf{u}_L, \mathbf{u}_R) \in \Omega^2 \to v_i^{\pm}(\mathbf{u}_L, \mathbf{u}_R) \in I\!R$ *are locally Lipschitz-continuous with the property that*

$$v_i^{\pm}(\mathbf{u}, \mathbf{u}) = u^{\pm} \quad \textit{for all} \quad \mathbf{u} \in \Omega.$$

Proof Invoking (3.61) and (3.62), equalities (3.63) readily imply

$$2\overline{\rho^{\gamma_i}}_j^{n+1=} \times (s_i)_j^{n+1=} = \overline{\rho^{\gamma_i}}_{j-1/2,R}^{n+1=} \times (s_i)_j^n - 2\frac{\Delta t}{\Delta x} w_i^{j-1/2,+}((s_i)_j^n - (s_i)_{j-1}^n)$$

$$+\overline{\rho^{\gamma_i}}_{j+1/2,L}^{n+1=} \times (s_i)_j^n - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,-}((s_i)_{j+1}^n - (s_i)_j^n),$$

that is to say, again thanks to (3.62)

$$(s_i)_j^{n+1=} = (s_i)_j^n - \frac{\Delta t}{\Delta x} \frac{w_i^{j-1/2,+}}{\overline{\rho^{\gamma_i}}_j^{n+1=}}((s_i)_j^n - (s_i)_{j-1}^n) - \frac{\Delta t}{\Delta x} \frac{w_i^{j+1/2,-}}{\overline{\rho^{\gamma_i}}_j^{n+1=}}((s_i)_{j+1}^n - (s_i)_j^n).$$

This concludes the proof.

The purpose of the following sections is to describe a relevant approximate Riemann solver which will yield a fairly simple evaluation of the discrete velocities defined in (3.67). As far as the exact Riemann solver is concerned, we may end up with somehow cumbersome formulae essentially because of the possible rarefaction waves. Our main objective is thus to suitably get rid of such waves when introducing approximate Riemann solutions systematically made of discontinuities separating constant states.

3.4 A Relaxation model

In order to meet the above requirement, we suggest to approximate the solutions of the system (3.14)-(3.15) by those of a suitable extended first order system with singular perturbation. Motivated by the work of Liu [101] and Chen, Levermore and Liu [28], such a system aims at restoring not only the original PDE model but also its entropy inequality in the regime of an infinite relaxation parameter. Here the extended first order system will have the property of being non linear hyperbolic with only linearly degenerate fields. Linear degeneracy is responsible for Riemann solutions uniquely made of discontinuities. Here and by contrast to the approach proposed by Jin and Xin [74], most of the non linearities in the original PDE model are kept for the sake of a better accuracy. Indeed, the relaxation system admits as natural solutions the contact discontinuities of the original PDE. Motivated by the work of Suliciu [131] (see also [32]), we suggest to suitably modify the total pressure law entering the original model since this law turns to concentrate all the genuine non linearities. The proposed approach has deep relationships with relaxation techniques already introduced in the literature, but is actually different in its design principle. Let us mention [32] in the 3×3 Euler setting and also [18] for a general approach devoted to flux vector splitting methods. Here we no longer consider the pressure as an independent variable equipped with its own PDE, but we keep this pressure as a non linear function of the unknown when suitably shadowing the exact role played by the specific volume τ. The strategy we propose actually considerably simplify the analysis of the relaxation model. The associated finite volumes method will actually turn to be algebraically equivalent to the ones described in [32], [18]. In the Lagrangian setting, the corresponding finite volumes method also coincides with the scheme proposed by Després [43] via another formalism (see also Gallice [51]). All these contributions can be understood as a fruitful reinterpretation of a scheme proposed by Toro [136] but which can be actually traced back to Viviand and Veuillot [144] in the mid seventies.

3.4.1 The model and its associated relaxation energy

We propose to approximate the entropy weak solutions of (3.14)-(3.15) by those of the following non linear first order system with singular perturbation :

$$\begin{cases} \partial_t \rho_\lambda + \partial_x (\rho u)_\lambda = 0, \\ \partial_t (\rho u)_\lambda + \partial_x (\rho u^2 + \Pi)_\lambda = 0, \\ \partial_t (\rho S_i)_\lambda + \partial_x (\rho S_i u)_\lambda = 0, \quad i = 1, ..., N, \\ \partial_t (\rho \mathcal{T})_\lambda + \partial_x (\rho \mathcal{T} u)_\lambda = \lambda \rho_\lambda (\tau_\lambda - \mathcal{T}_\lambda), \end{cases} \tag{3.68}$$

with the following closure equation :

$$\begin{aligned} \Pi = \Pi(\mathcal{T}, \{S_i\}_{1 \le i \le N}) = p(\mathcal{T}, \{S_i\}_{1 \le i \le N}) + a^2(\mathcal{T} - \tau), \quad \tau = \tfrac{1}{\rho}, \\ p(\mathcal{T}, \{S_i\}_{1 \le i \le N}) := \sum_{i=1}^{N} p_i(\mathcal{T}, S_i). \end{aligned} \tag{3.69}$$

From this definition, the equilibrium system (3.14) is clearly formally recovered in the regime of an infinite relaxation parameter $\lambda > 0$ since we get $\mathcal{T} = \tau$. In this loose sense, solutions of the relaxation system (3.68) may serve to approximate the solutions of the equilibrium system (3.14). But to prevent the proposed relaxation procedure from instabilities in the asymptotic regime $\lambda = +\infty$, it is well-known after the work by Liu [101] and Chen, Levermore and Liu [28], that the equilibrium system and the relaxation system must meet some compatibility conditions : eigenvalues of both systems must be properly interlaced. These so-called Whitham conditions will be seen hereafter to be satisfied for properly prescribed values of the positive real number a entering the definition of the relaxation pressure law (3.69).
Setting $\mathbf{U} =^T (\rho, \rho u, \{\rho S_i\}_{i=1,...,N}, \rho \mathcal{T})$, the relaxation system (3.68)–(3.69) is associated with the following natural phase space :

$$\Omega_{\mathbf{U}} = \{\mathbf{U} \in I\!R^{N+3}, \rho > 0,\ \rho S_i > 0,\ i = 1, ..., N,\ \rho \mathcal{T} > 0\}.$$

For simplicity in the notations, (3.68) will be given the next condensed form when $\lambda = 0$:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0, \tag{3.70}$$

where $\mathbf{F}$ finds an immediate definition. The first statement motivates the interest of the proposed relaxation procedure :

Proposition 17 *Let be given $a > 0$. Then the first order system (3.70) is hyperbolic for all $\mathbf{U}$ in $\Omega_{\mathbf{U}}$ with the following distinct eigenvalues :*

$$\lambda_1(\mathbf{U}) = u - a\tau = u - \frac{a}{\rho}, \quad \lambda_2(\mathbf{U}) = u, \quad \lambda_3(\mathbf{U}) = u + a\tau = u + \frac{a}{\rho}.$$

The eigenvalue $\lambda_2(\mathbf{U})$ has $(N+1)$ order of multiplicity while $\lambda_1(\mathbf{U})$ and $\lambda_3(\mathbf{U})$ are simple. All the fields under consideration are linearly degenerate.

As a consequence of the linear degeneracy of all the fields, the Riemann problem associated with (3.70) will be seen below to be trivially solvable. In addition, contact discontinuity solutions of the equilibrium system will be seen to be exactly preserved by the proposed relaxation procedure. These two highly desirable properties have actually dictated the derivation model (3.68)-(3.69).

Proof Easy calculations left to the reader give the required results. Here we simply highlight that the property of linear degeneracy of the field associated with the eigenvalues $\lambda_i(\mathbf{U})$ for $i = 1$ and $i = 3$ can only hold for sound speed $c = c(\rho, \{S_i\}_{i=1,\ldots,N})$ in the form $c = a/\rho$ where a is a positive real constant ($a = 0$ precludes the existence of a basis of eigenvectors). In that aim, observe that when considering general closure thermodynamics relations (*i.e.* for general sound speed given by $c^2 = -\partial_\tau p$), an eigenvector associated with the eigenvalue $\lambda_\epsilon(\mathbf{U}) = u + \epsilon c$ (with $\epsilon = -1$ or $\epsilon = 1$) is given by $r_\epsilon(u) = (1, \epsilon c/\rho, \{0\}_{i=1,\ldots,N})$. Then, asking $\nabla_{\mathbf{U}}\lambda_\epsilon(\mathbf{U}).r_\epsilon(\mathbf{U}) = \epsilon(\partial_\rho c + c/\rho) = 0$ to be valid for all $\mathbf{U}$ is nothing but the expected property.

The next statement proves that the relaxation system (3.68)–(3.69) enters the general framework proposed by Chen, Levermore and Liu [28] :

Theorem 7
Let us introduce the relaxation energy defined for all $\mathbf{U}$ in $\Omega_{\mathbf{U}}$ by :

$$\{\rho\Sigma\}(\mathbf{U}) = \frac{1}{2}\frac{(\rho u)^2}{\rho} + \rho\phi(\mathcal{T}, \{S_i\}_{1\le i\le N}), \tag{3.71}$$

where

$$\phi(\mathcal{T}, \{S_i\}_{1\le i\le N}) = \sum_{i=1}^{N}\varepsilon_i(\mathcal{T}, S_i) + \frac{1}{2a^2}\{\Pi^2(\mathcal{T}, \{S_i\}_{1\le i\le N}) - (\sum_{i=1}^{N} p_i(\mathcal{T}, S_i))^2\}. \tag{3.72}$$

This relaxation energy is consistent with the equilibrium energy in the sense that :

$$\rho\Sigma(\rho, \rho u, \{\rho S_i\}_{1\le i\le N}, \rho\tau) = \rho E, \quad \{(\rho\Sigma + \Pi)u\}(\rho, \rho u, \{\rho S_i\}_{1\le i\le N}, \rho\tau) = (\rho E + p)u.$$

In addition, smooth solutions of (3.68)–(3.69) satisfy the following energy equation for all $\lambda \ge 0$:

$$\partial_t\{\rho\Sigma\}(\mathbf{U}_\lambda) + \partial_x\{(\rho\Sigma + \Pi)u\}(\mathbf{U}_\lambda) = -\lambda\rho_\lambda(a^2 - \{\rho c\}^2(\mathbf{U}_\lambda))(\tau_\lambda - \mathcal{T}_\lambda)^2, \tag{3.73}$$

which yields an energy inequality under the following Whitham condition:

$$a^2 > \{\rho c\}^2(\mathcal{T}, \{S_i\}_{1\leq i\leq N}) = -\sum_{i=1}^{N}\{\partial_\tau p_i\}(\mathcal{T}, S_i), \tag{3.74}$$

for all the $\mathcal{T}$ and $\{S_i\}_{1\leq i\leq N}$ under consideration. Moreover, $(\rho,\ \rho u,\ \{\rho S_i\}_{1\leq i\leq N})$ being fixed, for all T-compact $\mathcal{K}$ including $\mathcal{T} = \tau$ so as to consider a finite value $a^2(\mathcal{K})$ in the Whitham condition (3.74), then the following minimization principle holds:

$$\rho E = \frac{1}{2}\frac{(\rho u)^2}{\rho} + \sum_{i=1}^{N}\rho\varepsilon_i(\tau, S_i) = \min_{\mathcal{T}\in\mathcal{K}}\{\{\rho\Sigma\}(\mathbf{U})\}. \tag{3.75}$$

In other words and under the Whitham condition (3.74), the relaxation energy spontaneously decreases in L^1 norm in the relaxation procedure. Moreover, its minimum coincides with the equilibrium energy ρE. With this respect, the relaxation energy plays the role of an entropy compatible with the relaxation procedure in the sense of Chen, Levermore and Liu [28]. According to the general framework proposed by these authors, one should have required the relaxation energy to be in addition strictly convex in $\mathbf{U}$. But here, such a convexity property is not useful since we do not need for a Lax entropy selection principle for the weak solutions of the PDE system (3.68)–(3.69). Indeed, all its fields are linearly degenerate and such a strong property is known (see [57], [118]) to imply that the equation (3.73), valid for smooth solutions, is still valid for weak solutions with equality in the sense of the distributions. The validity of (3.73) in $\mathcal{D}'$ together with the reported minimization principle will suffice for our forthcoming purposes.

Lemma 12 *Smooth solutions of (3.68)-(3.69) obey the following pressure-like equation for all $\lambda \geq 0$:*

$$\partial_t\Pi(\mathbf{U}_\lambda) + u_\lambda\partial_x\Pi(\mathbf{U}_\lambda) + a^2\tau_\lambda\partial_x u_\lambda = \lambda(a^2 - \{\rho c\}^2(\mathbf{U}_\lambda)) \times (\tau_\lambda - \mathcal{T}_\lambda), \tag{3.76}$$

where $\{\rho c\}^2(\mathbf{U})$ is defined with little abuse in the notation in (3.74).

Proof The validity of (3.76) stems from the property that smooth solutions of (3.68) satisfy the following transport equations:

$$\begin{cases} \partial_t S_i + u\partial_x S_i = 0, \quad i = 1, ..., N, \\ \partial_t\mathcal{T} + u\partial_x\mathcal{T} = \lambda(\tau - \mathcal{T}), \end{cases} \tag{3.77}$$

where the subscript λ has been (and will be from now on) omitted for simplicity in the notations. We thus easily infer the following identity:

$$\partial_t p(\mathcal{T}, \{S_i\}_{1\leq i\leq N}) + u\partial_t p(\mathcal{T}, \{S_i\}_{1\leq i\leq N}) = \lambda(\{\sum_{i=1}^{N}\partial_\tau p_i\}(\mathcal{T}, S_i))(\tau - \mathcal{T}). \tag{3.78}$$

Next and by the density equation, the specific volume τ is known to be solution of:

$$\partial_t\tau + u\partial_x\tau - \tau\partial_x u = 0,$$

so that we deduce:

$$\partial_t a^2(\mathcal{T} - \tau) + u\partial_x a^2(\mathcal{T} - \tau) + a^2\tau\partial_x u = \lambda a^2(\tau - \mathcal{T}). \tag{3.79}$$

The required pressure like equation (3.76) just follows from the definition (3.69) of $\Pi(\mathcal{T}, \{S_i\}_{1\leq i\leq N})$ when summing (3.78) and (3.79).

With this result, let us prove Theorem 7.

Proof Classical considerations imply from the density and momentum equations that smooth solutions obey the following equation for governing the kinetic energy :

$$\partial_t \rho \frac{u^2}{2} + \partial_x \rho \frac{u^3}{2} + u\partial_x \Pi = 0. \tag{3.80}$$

In order to derive an equation in conservation form, let us notice that the additional pressure-like equation (3.76) yields after easy manipulations :

$$\partial_t \rho \frac{\Pi^2}{2a^2} + \partial_x \rho u \frac{\Pi^2}{2a^2} + \Pi\partial_x u = \lambda\rho\Pi(1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}), \tag{3.81}$$

so as to arrive at :

$$\partial_t \rho\{\frac{u^2}{2} + \frac{\Pi^2}{2a^2}\} + \partial_x(\rho u\{\frac{u^2}{2} + \frac{\Pi^2}{2a^2}\} + \Pi u) = \lambda\rho\Pi(1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}). \tag{3.82}$$

The energy involved in (3.82) is obviously not consistent with the equilibrium total energy so that (3.82) must receive extra contributions as follows. Invoking again the transport equations (3.77), any given smooth nonlinear combination of $\mathcal{T}$ and the $\{S_i\}_{i=1,\dots,N}$, say

$$\theta(\mathcal{T}, \{S_i\}_{i=1,\dots,N}) = \sum_{i=1}^{N} \varepsilon_i(\mathcal{T}, S_i) - \frac{(\sum_{i=1}^{N} p_i(\mathcal{T}, S_i))^2}{2a^2},$$

is easily seen to satisfy :

$$\partial_t \rho\theta(\{S_i\}_{i=1,\dots,N}, I) + \partial_x \rho\theta(\{S_i\}_{i=1,\dots,N}, I)u = -\lambda\rho\sum_{i=1}^{N} p_i(\mathcal{T}, S_i)(1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}). \tag{3.83}$$

Therefore, summing (3.82) and (3.83) gives in view of the definition of the relaxation energy :

$$\partial_t \rho\Sigma + \partial_x(\rho\Sigma u + \Pi u) = \lambda\rho(1 - \frac{\{\rho c\}^2}{a^2}) \times (\Pi - \sum_{i=1}^{N} p_i(\mathcal{T}, S_i))(\tau - \mathcal{T}),$$

which is nothing but the required inequality (3.73) when invoking the definition (3.69) of the relaxation pressure law.
Let us now turn establishing the minimization principle (3.75). Let $(\rho, \rho u, \{\rho S_i\}_{1\leq i\leq N})$ being fixed and let us consider a fixed compact $\mathcal{K} = [\mathcal{T}_{min}, \mathcal{T}_{max}]$ with the property that $\mathcal{T} = \tau \in \mathcal{K}$. Then, easy calculations yield for all $\mathcal{T} \in \mathcal{K}$:

$$\{\partial_{\mathcal{T}} \rho\Sigma\}(\mathbf{U}) = \rho\partial_{\mathcal{T}} \phi(\mathcal{T}, \{S_i\}_{1\leq i\leq N}) = (a^2(\mathcal{K}) - \{\rho c\}^2(\mathcal{T}, \{S_i\}_{1\leq i\leq N})) \times (\mathcal{T} - \tau), \tag{3.84}$$

where $a^2(\mathcal{K})$ denotes some finite values satisfying the Whitham condition on $\mathcal{K}$. As a consequence, this first order derivative exactly vanishes once on $\mathcal{K}$: namely for $\mathcal{T} = \tau$. But the second $\mathcal{T}$-derivative evaluated at this solution gives:

$$\{\partial^2_{\mathcal{T}\mathcal{T}}\rho\Sigma\}(\rho, \rho u, \{\rho S_i\}_{1\leq i\leq N}, \rho\tau) = (a^2(\mathcal{K}) - \{\rho c\}^2(\tau, \{S_i\}_{1\leq i\leq N})) > 0, \tag{3.85}$$

again because of the Whitham condition. This suffices to prove that $\mathcal{T} = \tau$ is a local minimum on $\mathcal{K}$. Since the first derivative vanishes only once and this with a positive second derivative, the endpoints $\mathcal{T}_{min}$ and $\mathcal{T}_{max}$ of $\mathcal{K}$ cannot yield lower values of $\rho\Sigma$. Therefore the local minimum $\mathcal{T} = \tau$ is necessarily a global one. This completes the proof since by construction one has:

$$\{\rho\Sigma\}(\rho, \rho u, \{\rho S_i\}_{1\leq i\leq N}, \rho\tau) = \rho E.$$

Equipped with these stability results valid under the Whitham condition (3.74), we now turn describing a relevant finite volumes method for approximating the entropy weak solutions of the non linear hyperbolic system (3.14)-(3.15).

3.4.2 The numerical method

Assuming a piecewise constant approximate solution $\mathbf{u}_h$ to be given at time t^n, this one is evolved to the next time level $t^n + \Delta t$ when approximating the solution of the Cauchy problem (3.14)-(3.15) with initial data $\mathbf{u}_h(x, t^n)$ by the solution of the relaxation system (3.68) when prescribing its initial data $\mathbf{U}_h(x, t^n)$ at equilibrium from $\mathbf{u}_h(x, t^n)$ as follows:

$$\mathbf{U}_h(x, t^n) =^T (\rho_h(x, t^n), (\rho u)_h(x, t^n), \{(\rho S_i)_h(x, t^n)\}_{i=1,\ldots N}, (\rho\mathcal{T})_h(x, t^n)), \tag{3.86}$$

where by definition:

$$(\rho\mathcal{T})_h(x, t^n) = 1, \quad \text{for all} \quad x \in I\!R.$$

The precise choice of the constant a entering the definition of the relaxation pressure Π in (3.69) will be described later on. Its definition will be of course made to meet the Whitham condition (3.74) but some useful refinements will be performed under the convenient CFL restriction:

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{U}}(|u - a\tau|(\mathbf{U}), |u + a\tau|(\mathbf{U})) \leq \frac{1}{2}, \tag{3.87}$$

for all the $\mathbf{U}$ under consideration. Following Jin and Xin [74], we adopt the now classical two steps procedure when first solving the frozen relaxation problem: *i.e.* when choosing $\lambda = 0$, and when then projecting back its frozen solution onto the equilibrium manifold. Strictly speaking, this procedure consists in solving the following Cauchy problem for small times t (*i.e.* under the CFL condition (3.87)):

$$\left\{\begin{array}{l} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \Pi) = 0, \\ \partial_t(\rho S_i) + \partial_x(\rho S_i u) = 0, \quad i = 1, ..., N, \\ \partial_t(\rho\mathcal{T}) + \partial_x(\rho\mathcal{T} u) = 0, \end{array}\right. \tag{3.88}$$

with initial data $\mathbf{U}_h(x,t^n)$. Note again that no entropy selection principle is needed to select the weak solution $\mathbf{U}_h(x,t)$ under consideration since all the fields of (3.88) are linearly degenerate. Since $\lambda = 0$, this solution generally no longer belongs to the equilibrium manifold, and this motivates the following pointwise in x projection step at time $t^n + \Delta t$:

$$\mathcal{T}_h(x,t^n+\Delta t) = \tau_h(x,t^n+\Delta t), \quad \text{for all} \quad x \in I\!R. \tag{3.89}$$

Such an identity is formally equivalently obtained when solving the following ODE system with singular perturbation in the regime of an infinite relaxation parameter λ:

$$\begin{cases} \partial_t \rho_\lambda = 0, \\ \partial_t(\rho u)_\lambda = 0, \\ \partial_t(\rho S_i)_\lambda = 0, \quad i = 1,...,N, \\ \partial_t(\rho\mathcal{T})_\lambda = \lambda\rho_\lambda(\tau_\lambda - \mathcal{T}_\lambda). \end{cases} \tag{3.90}$$

This two steps procedure (actually a convenient variant of it) will be seen to yield a entropy consistent finite volumes method for approximating the weak solutions of (3.14)-(3.15) under a relevant choice of the parameter a.

3.4.3 Classical averagings in the relaxation approach

Let us now propose a precise description of the variant we suggest, motivated by an adaptative in space choice of a. A localized definition of the parameter a requires a brief discussion to prove that a relevant finite volumes method can be already derived in the setting of the classical averagings: note that both the density and the momentum have received in the previous section this type of averages. We shall conclude this section when exhibiting the exact Riemann solution for the Relaxation system. Its form will be used in the next section. To motivate an adaptive choice of a, let us observe that the relaxation energy inequality (3.73) with $\lambda > 0$ strongly suggests to choose the smallest value of a that satisfies the Whitham condition (3.74), in order to lower the numerical rate of dissipation measured here in terms of the discrete energy rate of dissipation. Here and after [39], we show that it is actually possible to optimize locally in space the choice of a under the CFL restriction (3.87). To that purpose, it is convenient to adopt a Harten, Lax and Van Leer type of analysis. Here, we suggest to define $\mathbf{U}_h(x,t)$ with $t \in (t^n, t^n+\Delta t)$ locally in space when setting:

$$\mathbf{U}_h(x,t) = \mathbf{U}_{a_{j+1/2}}\left(\frac{x - x_{j+1/2}}{t-t^n}; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n\right), \quad (x,t) \in (x_j, x_{j+1}) \times (t^n, t^n+\Delta t), \tag{3.91}$$

where the self-similar function $\mathbf{U}_{a_{j+1/2}}$ denotes the exact Riemann solution of (3.88) at interface $x_{j+1/2}$. Here the relaxation pressure law is locally defined by

$$\Pi = \Pi(\mathcal{T}, \{S_i\}_{1\le i\le N}) = p(\mathcal{T}, \{S_i\}_{1\le i\le N}) + a_{j+1/2}^2(\mathcal{T} - \tau),$$

for some relevant definition of $a_{j+1/2}$ given later on, but solely depending on the two states $\mathbf{U}_j^n$ and $\mathbf{U}_{j+1}^n$. This localized definition is obviously performed for the sake of minimizing the numerical dissipation. But consequently, the function $\mathbf{U}_h(x,t)$ we have defined along these lines is no longer the exact solution of the Cauchy problem since now a discontinuity takes place in the definition of the pressure law at each x_j.

Then the exact solution cannot be the superposition of the Riemann solutions $\mathbf{U}_{a_{j+1/2}}$. Nevertheless, the property that $\mathbf{U}_h(x,t^n)$ is at equilibrium (*i.e.* with $\mathcal{T}_j^n = \tau_j^n$ for all j) will allow us to check that, under the CFL condition (3.87), the proposed function $\mathbf{U}_h(x,t)$ still serves to define a conservative finite volumes method for approximating the entropy weak solutions of the equilibrium system (3.14)-(3.15). To prove this claim, let us consider the following half averagings of the function $\mathbf{U}_h(x,t)$:

$$\begin{aligned}\mathbf{U}^{n+1-}_{j+1/2,L} &= \tfrac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} \mathbf{U}_h(x,t^n+\Delta t)dx,\\ \mathbf{U}^{n+1-}_{j+1/2,R} &= \tfrac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}} \mathbf{U}_h(x,t^n+\Delta t)dx.\end{aligned}\tag{3.92}$$

We then have the following result :

Lemma 13 *Under the CFL condition (3.87), the averagings (3.92) meet the following equivalent form*

$$\begin{aligned}\mathbf{U}^{n+1-}_{j+1/2,L} &= \mathbf{U}^n_j - 2\tfrac{\Delta t}{\Delta x}(\mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}(0^+;\mathbf{U}^n_j,\mathbf{U}^n_{j+1})) - \mathbf{F}_{a_{j+1/2}}(\mathbf{U}^n_j)),\\ \mathbf{U}^{n+1-}_{j+1/2,R} &= \mathbf{U}^n_{j+1} - 2\tfrac{\Delta t}{\Delta x}(\mathbf{F}_{a_{j+1/2}}(\mathbf{U}^n_{j+1}) - \mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}(0^+;\mathbf{U}^n_j,\mathbf{U}^n_{j+1}))),\end{aligned}\tag{3.93}$$

where we have used the condensed notation $\mathbf{F}$ *introduced in (3.70). In addition, we have :*

$$\mathbf{F}_{a_{j+1/2}}(\mathbf{U}^n_j) = ((\rho u)^n_j, (\rho u^2+\Pi)^n_j, \{(\rho S_i u)^n_j\}_{i=1,\dots,N}, (\rho\mathcal{T}u)^n_j),\tag{3.94}$$

with

$$\Pi(\mathbf{U}^n_j) = \sum_{i=1}^{N}(p_i)^n_j, \quad \mathcal{T}(\mathbf{U}^n_j) = \tau^n_j.$$

The same property holds for $\mathbf{F}_{a_{j+1/2}}(\mathbf{U}^n_{j+1})$.
In other words, each of the fluxes $\mathbf{F}_{a_{j+1/2}}(\mathbf{U}^n_j)$ stays free from a particular choice of $a_{j+1/2}$.

Proof The restriction of the function $\mathbf{U}_h(x,t)$ to $(x_j,x_{j+1})\times(t^n,t^n+\Delta t)$ coincides with the exact Riemann solution of the Cauchy problem (3.88) at interface $x_{j+1/2}$. Thus by construction we have :

$$\partial_t \mathbf{U}_{a_{j+1/2}} + \partial_x \mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}) = 0, \quad (x,t)\in(x_j,x_{j+1})\times(t^n,t^n+\Delta t).$$

Classical considerations (see for instance [57]) then yield, with little abuse in the notation :

$$\frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} \mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)dx - \mathbf{U}^n_j + \frac{2}{\Delta x}(\Delta t\,\mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}(0^+;\mathbf{U}^n_j,\mathbf{U}^n_{j+1})) -$$
$$\int_{t^n}^{t^n+\Delta t}\mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}(\frac{x_j - x_{j+1/2}}{t-t^n};\mathbf{U}^n_j,\mathbf{U}^n_{j+1}))dt) = 0.$$

But in view of the CFL restriction (3.87) one clearly has :

$$\mathbf{U}_{a_{j+1/2}}(\frac{x_j - x_{j+1/2}}{t-t^n};\mathbf{U}^n_j,\mathbf{U}^n_{j+1}) = \mathbf{U}^n_j,$$

for all $t \in (t^n, t^n + \Delta t)$. The required first identity in (3.93) follows and the second one is derived along the same lines.
To conclude, it suffices to observe that by construction the states $\mathbf{U}_j^n$ are at equilibrium and hence the identity $\mathbf{F}_{a_{j+1/2}}(\mathbf{U}_j^n) = \mathbf{F}(\mathbf{U}_j^n)$ whatever $a_{j+1/2}$ is.
Equipped with this result and following [61], we are led to propose the following updating:

$$\mathbf{U}_j^{n+1-} = \frac{1}{2}(\mathbf{U}_{j+1/2,L}^{n+1-} + \mathbf{U}_{j-1/2,R}^{n+1-}), \tag{3.95}$$

so as to get the updating of the equilibrium approximate solution:

$$\mathbf{u}_j^{n+1} = \begin{cases} \rho_j^{n+1} = \rho_j^{n+1-}, \\ (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \\ (\rho S_i)_j^{n+1} = (\rho S_i)_j^{n+1-}, \quad i = 1, ..., N. \end{cases} \tag{3.96}$$

In agreement with the rough description of the algorithm, the method is advanced to the next time level when defining

$$\mathbf{U}_j^{n+1} = (\mathbf{u}_j^{n+1}, 1), \quad j \in Z.$$

Corollary 5 *Under the CFL restriction (3.87), the equilibrium approximate solution updates according to the following consistent finite volumes method:*

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\Delta \mathbf{f}_{j+1/2}^n, \tag{3.97}$$

where the numerical flux $\mathbf{f}_{j+1/2}^n$ is given from the first $(N+2)$ components of the next Lipschitz-continuous and consistent numerical flux function:

$$(\mathbf{f}_{j+1/2}^n, .) = (\mathbf{F}_{a_{j+1/2}}(\mathbf{U}_{a_{j+1/2}}(0^+; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)). \tag{3.98}$$

The relevance of the proposed finite volumes method will be assessed in the next paragraph when properly prescribing the value of the parameter a at each interface $x_{j+1/2}$ so as to meet the Whitham condition (3.74). Let us conclude the present one when deriving a fairly simple form of expression for the numerical flux function (3.98). The simplicity in the evaluation of this flux comes from the property that the Riemann problem for (3.88) is trivially solved according to:
Lemma 14 *Let be given $\mathbf{U}_L$ and $\mathbf{U}_R$ two states in $\Omega_{\mathbf{U}}$ at equilibrium (i.e $\mathcal{T}_L = \tau_L$ and $\mathcal{T}_R = \tau_R$). Let us then define $a := a(\mathbf{U}_L, \mathbf{U}_R) > 0$, the parameter entering the relaxation pressure law (3.69), large enough such that the following natural ordering holds:*

$$\lambda_1(\mathbf{U}_L) = u_L - a\tau_L < u^\star < \lambda_3(\mathbf{U}_R) = u_R + a\tau_R, \tag{3.99}$$

where $u^\star$ stands for some intermediate velocity given by:

$$u^\star = \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(p_R - p_L).$$

Then, the self-similar solution $\mathbf{U}_a(x/t; \mathbf{U}_L, \mathbf{U}_R)$ of the Cauchy problem (3.88) for the initial data

$$\mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L & \textit{if} \quad x < 0, \\ \mathbf{U}_R & \textit{if} \quad x > 0, \end{cases} \tag{3.100}$$

is systematically made of at most four constant states separated by three discontinuities as follows:

$$\mathbf{U}(x/t;\mathbf{U}_L,\mathbf{U}_R)=\begin{cases}\mathbf{U}_L & \text{if } \frac{x}{t}<\lambda_1(\mathbf{U}_L),\\ \mathbf{U}_1 & \text{if } \lambda_1(\mathbf{U}_L)<\frac{x}{t}<\lambda_2(\mathbf{U}_1),\\ \mathbf{U}_2 & \text{if } \lambda_2(\mathbf{U}_2)<\frac{x}{t}<\lambda_3(\mathbf{U}_R),\\ \mathbf{U}_R & \text{if } \lambda_3(\mathbf{U}_R)<\frac{x}{t},\end{cases}$$

with

$$\lambda_2(\mathbf{U}_1)=\lambda_2(\mathbf{U}_2)=u^\star.$$

The intermediate states in this Riemann solution are defined from :

$$\begin{array}{l} u_1=u_2=u^\star,\\ (S_i)_1=(S_i)_L, \quad 1\le i\le N,\\ (S_i)_2=(S_i)_R, \quad 1\le i\le N,\\ \mathcal{T}_1=\tau_L,\\ \mathcal{T}_2=\tau_R, \end{array}$$

together with

$$\rho_1=a/(u^\star-\lambda_1(\mathbf{U}_L)), \quad \rho_2=a/(\lambda_3(\mathbf{U}_R)-u^\star),$$

so that under the ordering condition (3.99), both ρ_1 and ρ_2 are positive. As a consequence of these formulae, the intermediate states $\mathbf{U}_1$ and $\mathbf{U}_2$ are in $\Omega_{\mathbf{U}}$, and are associated with the following total pressure:

$$\Pi(\mathbf{U}_1)=p_L+a^2(\tau_L-\tau_1)=p_R+a^2(\tau_R-\tau_2)=\Pi(\mathbf{U}_2).$$

Proof Since all the fields are linearly degenerate, the Riemann solution we seek for is uniquely made of discontinuities, the i^{th} one propagating at the characteristic speed λ_i of the field under consideration. These discontinuities separate at most four constant states $\mathbf{U}_L$, $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_R$, two neighbouring states being solutions of the Rankine-Hugoniot conditions associated with the system (3.88). Once the parameter a obeys the natural ordering condition (3.99) in the three wave velocities, these jump conditions are easily seen to yield the conclusion.

3.4.4 The total energy stability estimate

Let us now address the relevant definition of the parameter $a_{j+1/2}$ entering the numerical flux function (3.98) at each interface $x_{j+1/2}$. To this purpose, let us define from the Riemann solution $\mathbf{U}_{a_{j+1/2}}(.;\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)$ the following self-similar function:

$$\mathbf{u}_{a_{j+1/2}}(x,t):={}^T(\rho,\rho u,\{\rho S_i\}_{i=1,\dots,N})_{a_{j+1/2}}(\frac{x-x_{j+1/2}}{t-t^n}),$$
$$(x,t)\in(x_j,x_{j+1})\times(t^n,t^n+\Delta t).$$

Observe that the following half averagings:

$$\begin{aligned} \mathbf{u}_{j+1/2,L}^{n+1} &= \tfrac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}}\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)dx,\\ \mathbf{u}_{j+1/2,R}^{n+1} &= \tfrac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}}\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)dx, \end{aligned} \tag{3.101}$$

just restore the proposed update for the equilibrium solution $\mathbf{u}_j^{n+1}$ from the next identity:

$$\mathbf{u}_j^{n+1} = \frac{1}{2}(\mathbf{u}_{j+1/2,L}^{n+1} + \mathbf{u}_{j-1/2,R}^{n+1}). \tag{3.102}$$

Equipped with this suitable function, we now state the main result of this section:

Theorem 8 *Let us assume that $a_{j+1/2} := a_{j+1/2}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ is chosen large enough so as to satisfy the ordering condition (3.99) together with the following Whitham conditions:*

$$\begin{aligned}
&\sum_{i=1}^{N} \tfrac{\partial p_i}{\partial \tau}(\tau_j^n, (S_i)_j^n) + a_{j+1/2}^2 > 0, \quad \sum_{i=1}^{N} \tfrac{\partial p_i}{\partial \tau}((\tau_1)_{j+1/2}^n, (S_i)_j^n) + a_{j+1/2}^2 > 0,\\
&\sum_{i=1}^{N} \tfrac{\partial p_i}{\partial \tau}(\tau_{j+1}^n, (S_i)_{j+1}^n) + a_{j+1/2}^2 > 0, \quad \sum_{i=1}^{N} \tfrac{\partial p_i}{\partial \tau}((\tau_2)_{j+1/2}^n, (S_i)_{j+1}^n) + a_{j+1/2}^2 > 0,
\end{aligned} \tag{3.103}$$

where $(\tau_1)_{j+1/2}^n$ and $(\tau_2)_{j+1/2}^n$ denote the specific volumes in the intermediate states of the Riemann solution $\mathbf{U}_{a_{j+1/2}}(., \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$. Then, under the CFL restriction (3.87), the following energy-like inequalities hold:

$$\begin{aligned}
&\frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} \{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t))dx \le (\rho E)_j^n - 2\frac{\Delta t}{\Delta x}(\{(\rho\Sigma + \Pi)u\}_{j+1/2}^n - (\rho H u)_j^n),\\
&\frac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}} \{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t))dx \le (\rho E)_{j+1}^n - 2\frac{\Delta t}{\Delta x}((\rho H u)_{j+1}^n - \{(\rho\Sigma + \Pi)u\}_{j+1/2}^n),
\end{aligned} \tag{3.104}$$

where by definition,

$$\{(\rho\Sigma + \Pi)u\}_{j+1/2}^n = \{(\rho\Sigma + \Pi)u\}(\mathbf{U}_{a_{j+1/2}}(0^+; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)).$$

Observe that for sufficiently large values of a, $(\tau_1)_{j+1/2}^n$ (respectively $(\tau_2)_{j+1/2}^n$) becomes arbitrarily close to τ_j^n (respectively τ_{j+1}^n). As an immediate consequence, it is easy to check that if a is chosen large enough, the Whitham conditions (3.99) (as well as the ordering property (3.103)) necessarily hold. We refer the reader to Bouchut [18] for simplified Whitham conditions which are more convenient to deal with from a numerical point of view.

Proof We prove below the first inequality in (3.104). The companion one will follow from simple adaptations. Let us first recall that the Riemann solution $\mathbf{U}_{a_{j+1/2}}(.; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ is obtained when choosing $\lambda = 0$ in the relaxation procedure. So that it obeys in the sense of the distributions the following total energy additional conservation law in view of Theorem 7:

$$\partial_t\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}) + \partial_x\{(\rho\Sigma + \Pi)u\}(\mathbf{U}_{a_{j+1/2}}) = 0.$$

Let us indeed recall that the property of linear degeneracy of all the fields implies the validity of the above conservation law not only for smooth but also weak solutions of

(3.88). Now classical arguments give:

$$\frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} \{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t))dx =$$

$$(\rho E)_j^n - \frac{2}{\Delta x}(\Delta t\{(\rho\Sigma+\Pi)u\}_{j+1/2}^n - \int_{t^n}^{t^n+\Delta t}\{(\rho\Sigma+\Pi)u\}(\mathbf{U}_{a_{j+1/2}}(x_j,t))dt). \tag{3.105}$$

But exactly the same arguments as those developed in the proof of Lemma 13 apply to prove that under the CFL restriction (3.87):

$$\int_{t^n}^{t^n+\Delta t}\{(\rho\Sigma+\Pi)u\}(\mathbf{U}_{a_{j+1/2}}(x_j,t))dt = \Delta t((\rho\Sigma+\Pi)u)_j^n = \Delta t(\rho H u)_j^n,$$

since and again $\mathbf{U}_j^n$ is by construction at equilibrium.
Let us next prove that the following inequality for all $x \in (x_j, x_{j+1})$:

$$\rho E(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)) \leq \rho\Sigma(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)). \tag{3.106}$$

This inequality will then easily imply the required entropy-like inequality from (3.105). To prove (3.106), we first observe that for all $x \in (x_j, x_{j+1/2}+\lambda_1(\mathbf{U}_j^n)\Delta t)$, one trivially has:

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)) = \{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)) = (\rho E)_j^n.$$

Let us now prove that for all x in $\mathcal{I}_1 = (x_{j+1/2}+\lambda_1(\mathbf{U}_j^n)\Delta t, x_{j+1/2}+u^\star\Delta t)$, namely when $\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t) = (\mathbf{U}_1)_{j+1/2}^n$, that

$$\rho E(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)) \leq \rho\Sigma(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)). \tag{3.107}$$

To that purpose, let us define the following compact $\mathcal{K} = [\mathcal{T}_{min}, \mathcal{T}_{max}]$ where $\mathcal{T}_{min} = \min(\tau_j^n, (\tau_1)_{j+1/2}^n)$ and $\mathcal{T}_{max} = \max(\tau_j^n, (\tau_1)_{j+1/2}^n)$, so that both $\mathcal{T}(x,t^n+\Delta t) = \tau_j^n$ and $\tau(x,t^n+\Delta t) = (\tau_1)_{j+1/2}^n$ in the Riemann solution belongs to this compact for all $x \in \mathcal{I}_1$. But, the first two discrete Whitham conditions just express that $a_{j+1/2}^2$ satisfies the requirement for applying the minimization principle stated in Theorem 7 over the compact $\mathcal{K}$ for any given fixed $x \in \mathcal{I}_1$. This indeed follows from the convexity of each of the partial pressure laws in τ, which ensures the monotonicity of $\tau \to \partial_\tau \sum_{i=1}^N p_i$. The validity of (3.107) thus follows. Exactly in the same way, one can prove that:

$$\rho E(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)) \leq \rho\Sigma(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)),$$

for all $x \in (x_{j+1/2}+u^\star\Delta t, x_{j+1/2}+\lambda_3(\mathbf{U}_{j+1}^n)\Delta t)$ while:

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t)) = \{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)) = (\rho E)_{j+1}^n,$$

when $x \in (x_{j+1/2}+\lambda_3(\mathbf{U}_{j+1}^n)\Delta t, x_{j+1})$. This completes the proof.

3.4.5 The new averaging procedure in the relaxation approach

This section aims at briefly exemplifying the averaging techniques introduced in Definition 4 within the frame of the present relaxation procedure. Let us from now on notice that the Riemann solutions associated with the relaxation system share with the exact equilibrium Riemann solutions the property that the specific entropies are just the superposition of step functions, each propagating with the speed of a contact discontinuity. Consequently, the representation formula (3.65) applies and exactly the same definitions (3.67) of the discrete velocities $v_i^{j+1/2,\pm}$ follow to get the next updating formulae :

$$(s_i)_j^{n+1=} = (s_i)_j^n - \frac{\Delta t}{\Delta x} v_i^{j-1/2,+}((s_i)_j^n - (s_i)_{j-1}^n) - \frac{\Delta t}{\Delta x} v_i^{j+1/2,-}((s_i)_{j+1}^n - (s_i)_j^n). \quad (3.108)$$

Here the very benefit comes from the very simple structure of each of the Riemann solutions which makes fairly straightforward the evaluation of

$$v_i^{j+1/2,\pm} = v_i^{\pm}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = \int_0^{(u_{j+1/2}^\star)^\pm} \rho^{\gamma_i}(\xi) d\xi / \overline{\rho^{\gamma_i}}_j^{n+1=}. \quad (3.109)$$

The relevance of the Relaxation approach with respect to this new averaging technique is assessed by this last statement :

Proposition 18 *Under the CFL restriction (3.87), the following energy-like inequalities hold :*

$$\{\rho E\}(\mathbf{u}_j^{n+1=}) - (\rho E)_j^n + \lambda \Delta \{\rho \mathcal{H} u\}_{j+1/2}^n \leq 0, \quad j \in Z, \quad (3.110)$$

where with the notations of the previous section, the numerical energy flux reads :

$$\{\rho \mathcal{H} u\}_{j+1/2}^n = \{(\rho \Sigma + \Pi) u\}(\mathbf{U}_{a_{j+1/2}}(0^+; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)).$$

Proof As an immediate consequence of the inequalities (3.104), we get under the CFL restriction (3.87) :

$$\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(\overline{\mathbf{w}}(x, t^n + \Delta t)) dx \leq (\rho E)_j^n - \frac{\Delta t}{\Delta x} \Delta \{\rho \mathcal{H} u\}_{j+1/2}^n,$$

where we have set :

$$\overline{\mathbf{w}}(x, t^n + \Delta t) = \begin{cases} \mathbf{u}_{a_{j-1/2}}(x, t^n + \Delta t), & x \in (x_{j-1/2}, x_j), \\ \mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t), & x \in (x_j, x_{j+1/2}). \end{cases}$$

Recalling the identities (3.101)-(3.102), we have thus arrived at exactly the same form of inequality (3.43) which is the starting point of the proof of the Lemma 11. The conclusion follows when applying exactly the same steps.

Observe that after the correction step coming with the present relaxation approach, the specific entropies receive exactly the same formula as the one proposed in (3.54). The positiveness of these corrected specific entropies is here again ensured by the energy inequalities (3.110). To conclude, let us emphasize that Corollary 4 applies to the present method exactly in the same lines.

3.5 Numerical experiments

This section aims at illustrating that the new averaging procedures we have described succeed in capturing numerical solutions of (3.1) in fairly good agreement with exact ones. We consider constant viscosity laws $\{\mu_i\}_{i=1,\dots,N}$ with a Reynolds number equal to 10^5 which is typical in aerodynamic. The numerical experiments we propose consist in two computations performed when considering Riemann initial data:

$$\mathbf{V}(x,0) = \begin{cases} \mathbf{V}_L & \text{if} \quad x < 0, \\ \mathbf{V}_R & \text{if} \quad x > 0, \end{cases} \tag{3.111}$$

for some given $\mathbf{V}_L$ and $\mathbf{V}_R$ in the phase space Ω, and when dealing with an uniform mesh made of 300 points. We then compare the corresponding numerical solutions together with the exact ones. To assess the importance of the correction step, we also display the approximate solutions obtained by a classical splitting approach (without correction procedure).

Experiment 1

We set $N = 3$ and choose

γ_1	γ_2	γ_3	μ_2/μ_1	μ_3/μ_1
1.4	1.6	1.4	1.	1.

while the left and right states entering (3.111) write as follows:

Experiment 1

	density	velocity	pressure 1	pressure 2	pressure 3
$\mathbf{V}_L$	3.	2.	1.5	1.	1.

	density	velocity	pressure 1	pressure 2	pressure 3
$\mathbf{V}_R$	2.6848	-2.1586	1.1930	0.7086	0.7649

Let us observe that despite the exact solution exhibits large pressure jumps, the numerical solution produced by our scheme fully agrees with the exact one, contrary to the approximate solution given by the classical splitting approach.

Experiment 2

We now deal with ratios of the viscosity laws $\{\mu_i/\mu_1\}_{i=2,3}$ large with respect to 1. More precisely, we consider:

γ_1	γ_2	γ_3	μ_2/μ_1	μ_3/μ_1
1.4	1.6	1.6	100.	100.

The left and right states entering (3.111) read as follows:

Experiment 2

$\mathbf{V}_L$

density	velocity	pressure 1	pressure 2	pressure 3
1.	1.	1.	1.	0.6

$\mathbf{V}_R$

density	velocity	pressure 1	pressure 2	pressure 3
1.4291	-0.5477	0.2277	0.4355	0.0929

Once more, let us notice the excellent agreement between the solutions given by our procedure and the exact ones.

Experiment 1

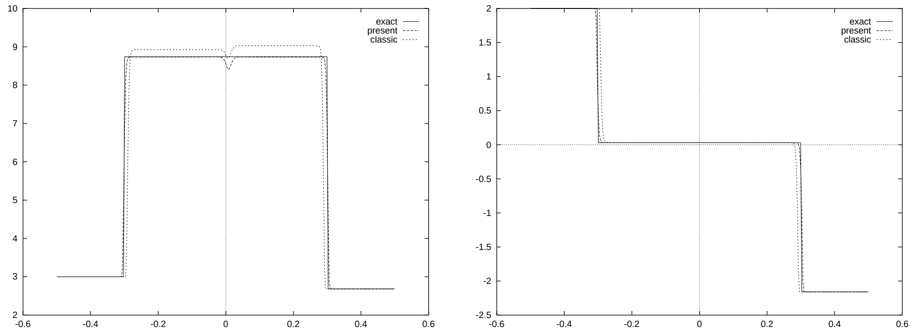

FIG. 3.1 – *Density, Velocity*

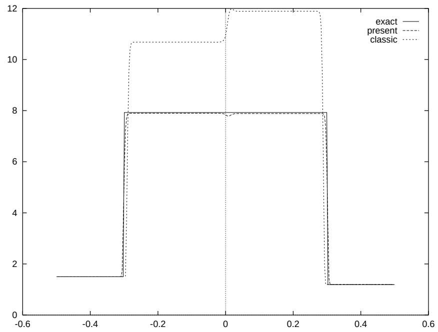

FIG. 3.2 – *Pressure 1*

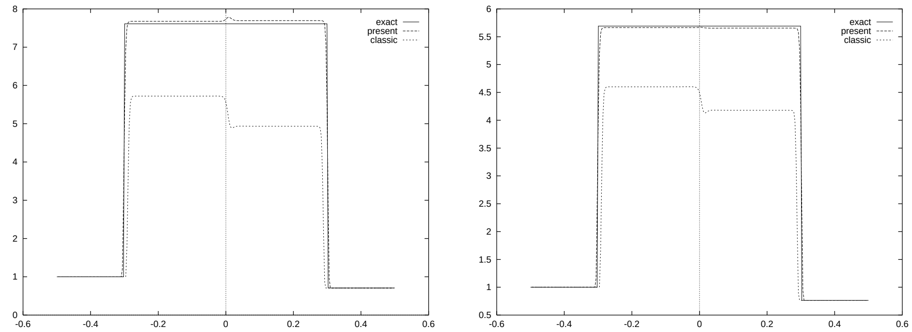

FIG. 3.3 – *Pressure 2, Pressure 3*

Experiment 2

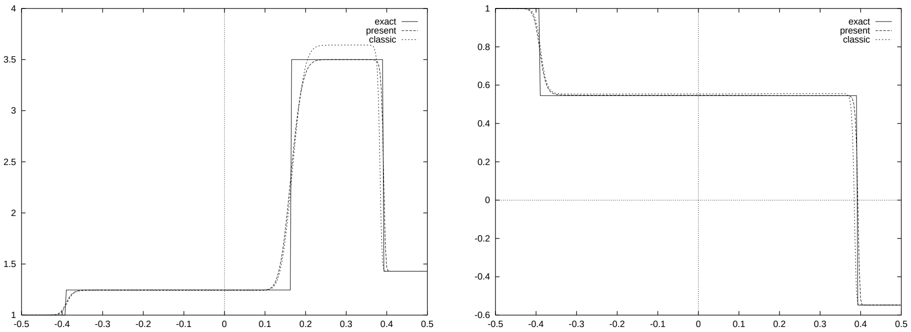

FIG. 3.4 – *Density, Velocity*

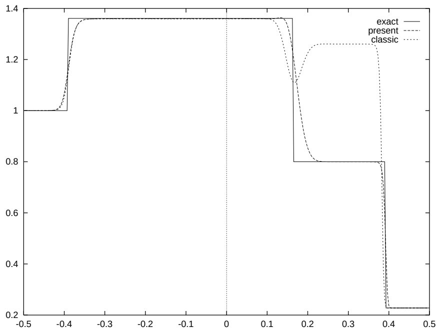

FIG. 3.5 – *Pressure 1*

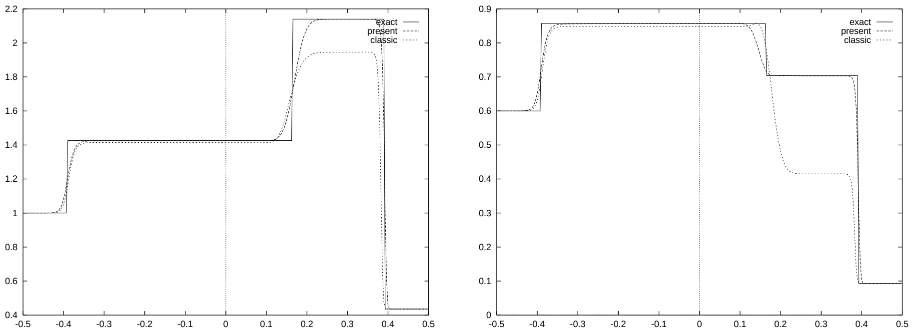

FIG. 3.6 – *Pressure 2, Pressure 3*