TIME-IMPLICIT APPROXIMATION OF THE MULTI-PRESSURE GAS DYNAMICS EQUATIONS IN SEVERAL SPACE DIMENSIONS

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Abstract. The present work is devoted to the numerical approximation of the solutions of the inviscid limit of multi-pressure Navier-Stokes (NS) equations in several space dimensions. The nonconservative form of the Euler-like limit model makes the shock solutions to be sensitive with respect to the underlying small scales, and then challenging their numerical approximation. In particular, classical algorithms fails in producing good numerical results. Here we are mainly concerned with (large time stepping) *implicit* numerical strategies. We first exhibit a set of generalized jump conditions satisfied by the shock solutions and well-suited to derive a time-implicit scheme. We then devise a linearized time-implicit solver for the sake of efficiency. This solver is shown to preserve the positivity of each internal energy ϵ_i provided that the total internal energy stays positive.

Key words. Multi-pressure Navier-Stokes equations, nonconservative system, generalized Rankine Hugoniot conditions, finite volume methods, time-implicit numerical scheme

1. Introduction. In this work, we are interested in the numerical approximation of the solutions of the following convective-diffusive system *in non-conservation form* :

$$\left\{\begin{array}{l} \partial_t \rho + \nabla.\ \rho \mathbf{w} = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ \partial_t \rho \mathbf{w} + \nabla.\ (\rho \mathbf{w} \otimes \mathbf{w} + \sum_{i=1}^{N} p_i(\rho, \rho\epsilon_i)\ \mathbf{I_d}) = \varepsilon\ \nabla.\ \big(\sum_{i=1}^{N} \mu_i(\rho, \rho\epsilon_i)\ \sigma\big), \\ \partial_t \rho\epsilon_i + \nabla.\ \rho\epsilon_i \mathbf{w} + p_i(\rho, \rho\epsilon_i) \nabla.\ \mathbf{w} = \varepsilon\ \mu_i(\rho, \rho\epsilon_i)\ \sigma : \nabla \mathbf{w},\ i = 1, ..., N, \end{array}\right. \tag{1.1}$$

with $\sigma = (\nabla \mathbf{w} + {}^t\nabla \mathbf{w}) - \frac{2}{3} \nabla.\ \mathbf{w}\ \mathbf{I_d}$. Here, the small parameter $\varepsilon > 0$ denotes the inverse of a Reynolds number. The main emphasis is put on the solutions of (1.1) in the inviscid regime, namely in the limit $\varepsilon \to 0^+$. With classical notations, ρ denotes the density of a complex compressible material with velocity $\mathbf{w} \in \mathbb{R}^d$ and modelled by N independent internal energies $\{\rho\epsilon_i\}_{i=1,...,N}$. The above system is equivalent to the usual Navier-Stokes equations in conservation form in the case of a single internal energy $N = 1$, and consists of a natural extension for larger values of N. Here, $N \geq 2$ independent pressure laws $p_i(\rho, \rho\epsilon_i)$ are present and the N associated internal energies $\rho\epsilon_i$ are given an independent evolution equation. We highlight right now that the proposed model does not rewrite in general in conservation form. For $i = 1, \ldots, N$, $\mu_i \geq 0$ represents the (smooth) viscosity law associated with the internal energy $\rho\epsilon_i$. We shall assume that $\mu := \sum_{i=1}^{N} \mu_i > 0$.

Several models from the physics enter the present framework. Let us quote for instance compressible turbulence models like the $k - \epsilon$ model (see [5], [3]), and more generally the multi-scale models (see [11]) for which a laminar pressure and some turbulent pressures are involved, or even models coming from the plasma physics (see [23]).

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The PDE model under consideration can be given the following condensed form:

$$\partial_t \mathbf{V}^\epsilon + \sum_{k=1}^{d} \mathcal{A}_k(\mathbf{V}^\epsilon)\ \partial_{x_k}\mathbf{V}^\epsilon = \varepsilon\ \mathcal{B}(\mathbf{V}^\epsilon,\ \sum_{k,l=1}^{d} \partial_{x_k}\mathcal{D}_k^l(\mathbf{V}^\epsilon)\partial_{x_l}\mathbf{V}^\epsilon). \tag{1.2}$$

The underlying nonlinear first-order system will be seen to be hyperbolic so that its solutions may develop discontinuities in finite time. Due to the non-conservation form of (1.2), the classical weak theory of distributions cannot cope with the inviscid limit $\varepsilon \to 0^+$. Several theories have been developed over the past two decades to handle this singular limit. We refer to the pionneering works by LeFloch [34], Dal Maso, LeFloch and Murat [25], and also by Colombeau and Leroux [21]. Note that these contributions specifically deal with first order nonconservative products, assuming that the viscous perturbations stand in conservation form. In the present work, we use a distinct mathematical framework specifically devoted to treat viscous perturbations in non-conservation form. Let us stress at this stage that all these theories share the same challenging issue : the discontinuous solutions of the inviscid limit equations are regularization-dependent. Therefore, they provide relevant and complementary tools to account for the sensitiveness of the shock solutions with respect to the underlying viscous small scales.

The theoretical framework we consider hereafter specifically assumes the existence of a change of variable $\mathbf{V} \to \mathbf{v}(\mathbf{V})$ so that (1.2) rewites:

$$\partial_t \mathbf{v}^\epsilon + \sum_{k=1}^{d} \partial_{x_k}\mathbf{F}_k(\mathbf{v}^\epsilon) = \varepsilon\ \mathcal{R}(\mathbf{v}^\epsilon,\ \sum_{k,l=1}^{d} \partial_{x_k}\mathcal{D}_k^l(\mathbf{v}^\epsilon)\partial_{x_l}\mathbf{v}^\epsilon). \tag{1.3}$$

In other words, the viscous perturbation still reads in non-conservation form but the first order terms now stand in divergence form. Let us now consider the inviscid limit $\varepsilon \to 0^+$ in the equivalent system (1.3). We stress that we cannot expect the viscous perturbation to identically vanish everywhere due to its nonconservative nature. Instead, we get a vector-valued bounded Borel measure $\kappa_{\mathbf{v}}$ concentrated in the shocks of limit solutions:

$$\partial_t \mathbf{v} + \nabla.\ \mathbf{F}(\mathbf{v}) = \kappa_{\mathbf{v}}. \tag{1.4}$$

This system will be given a rigorous mathematical formulation hereafter on the ground of travelling waves solutions to (1.3). Such a strategy has already been used to prove existence and uniqueness of Riemann solutions to the inviscid limit of (1.1) in one space variable (see [15]). The right hand side in (1.4) naturally gives rise to generalized jump conditions and is refered as to kinetic relations. This terminology is taken from other parts of the Physics of complex compressible material. Let us quote for instance the theory of undercompressive discontinuous solutions arising in the theory for phase transitions (see [1], [33]).

From a numerical standpoint, the sensitiveness of shock solutions with respect to the viscous small scales makes particularly difficult the capture of the discontinuous solutions in (1.4). Indeed, discrete shock solutions turn to be deeply dependent on the numerical diffusion of the scheme and the latter does not match generally speaking with the exact viscous mechanisms in (1.3). This leads to large errors between exact and discrete solutions. Let us stress that even the original Godunov method fails in producing good numerical results. This failure has been analyzed and cured in several contributions by Berthon-Coquel [2], [4], Chalons-Coquel [12], [13], [14], [16].

The proposed analysis suggested several correction procedures. These techniques have been devised in the setting of fully explicit methods.
By contrast, the present work is concerned with the capture of stationary discontinuous solutions of (1.4) in several space dimensions. Achieving such a goal with efficiency requires the development of large time stepping implicit strategies. To meet this requirement, we propose a new correction procedure well-adapted to large CFL stabilizing techniques. The proposed method takes the form of a predictor-corrector strategy built for efficiency from a linearized time implicit solver. This method is shown to preserve the positivity of each internal energy ϵ_i, $1 \leq i \leq N$, provided that the sum of all the internal energies stays positive. Note that a complete stability theory is first obtained in the usual setting for time-explicit method. The key findings in this framework stay at the very basis of the extensions to large time stepping formulations. Numerical evidences in several space dimensions highlight the relevance of the method, assessing in particular its capability to produce perfectly stationary solutions.

To conclude the first part of the introduction, we mention that a longer version of this paper is proposed in [18]. It contains the details of most of the proofs of the results stated here.

1.1. Closure relations and first properties. Let us first consider the closure equations defining the N pressure laws $p_i = p_i(\rho, \rho\epsilon_i)$. With a little abuse in the notations, we will also write $p_i = p_i(\tau, \epsilon_i)$, $\tau = 1/\rho$. According to the second principle of thermodynamics, we shall assume that the relation

$$-T_i(\tau, \epsilon_i)dS_i = d\epsilon_i + p_i(\tau, \epsilon_i)d\tau \tag{1.5}$$

defines a smooth and strictly convex mapping $(\tau, \epsilon_i) \to S_i(\tau, \epsilon_i) > 0$ for any given $i = 1, .., N$, where the temperature $T_i(\tau, \epsilon_i)$ is assumed to stay positive. As a well-known consequence (see [27] for instance), the well-defined mapping $(\tau, S_i) \to \varepsilon_i(\tau, S_i)$ is strictly convex, and so does the mapping $(\rho, \rho\varepsilon_i) \to \rho S_i(\rho, \rho\varepsilon_i) := \rho S_i(1/\rho, \rho\varepsilon_i/\rho)$ and $(\rho, \rho S_i) \to \rho\varepsilon_i(\rho, \rho S_i) := \rho\varepsilon_i(1/\rho, \rho S_i/\rho)$ (here again with some little abuse in the notations).
In addition, each energy law is assumed to obey the Weyl's assumptions (see again [27]) and in particular the asymptotic conditions $\lim_{\rho S_i \to 0^+} \rho\varepsilon_i(\rho, \rho S_i) = +\infty$ and $\lim_{\rho S_i \to +\infty} \rho\varepsilon_i(\rho, \rho S_i) = 0$. Note that all the above assumptions are quite classical in the frame of the usual Navier-Stokes equations ($N = 1$).

Owing to these assumptions, one can easily prove the following statement that highlights the relationships with the usual setting $N = 1$. Let us define

$$\Omega = \{\mathbf{V} := (\rho, \rho\mathbf{w}, \{\rho\epsilon_i\}_{1\leq i\leq N}) \in \mathbb{R}^{N+d+1}/\rho > 0,\ \rho\mathbf{w} \in \mathbb{R}^d,\ \rho\epsilon_i > 0,\ 1 \leq i \leq N\}.$$

LEMMA 1.1. *Let $\mathbf{n} = {}^t(n_1, ..., n_d)$ be a unit vector in $\mathbb{R}^d$. The first order system extracted from (1.1) is hyperbolic over Ω, with the following eigenvalues : $\lambda_1(\mathbf{V}) = \mathbf{w}.\mathbf{n} - c \leq \lambda_{j=2,...,N+d}(\mathbf{V}) = \mathbf{w}.\mathbf{n} \leq \lambda_{N+d+1}(\mathbf{V}) = \mathbf{w}.\mathbf{n} + c$, with $c^2(\mathbf{V}) = \sum_{i=1}^N c_i^2(\mathbf{V})$ and where each partial sound speed is such that $c_i^2(\mathbf{V}) := (\partial_\rho p_i)_{\rho S_i} > 0$. The $1-$ and $(N+d+1)-$ characteristic fields are genuinely nonlinear, all the other characteristic fields being linearly degenerate.*
In other words, the intermediate fields are associated with contact discontinuities in the $\mathbf{n}$-direction and across which the eigenvalue $\mathbf{w}.\mathbf{n}$ stays continuous. As a consequence, these fields do not induce ambiguity in the nonconservative products

$p_i(\rho,\rho\epsilon_i)\nabla.\ \mathbf{w}$ if we focus on the first order system extracted from (1.1). This is already the case when $N=1$.
By contrast, the two extreme characteristic fields are genuinely nonlinear and thus responsible for the occurence of shock waves in the $\mathbf{n}$-direction and where the velocity $\mathbf{w}.\mathbf{n}$ and the partial pressure laws $p_i(\rho,\rho\epsilon_i)$ have nontrivial jumps. Hence, ambiguities are in order in the nonconservative products entering the first order system extracted from (1.1). This is of course not the case when $N=1$.

The non-conservation form of (1.1) thus rises the question of the existence of an admissible change of variables that would recast (1.1) in full conservation form. With this respect, the next result gives all the additional nontrivial equations satisfied by the smooth solutions of (1.1) and having (at least) a convective part in conservation form (see [13] or [2], [5] when $N=2$) :

THEOREM 1.2. *Smooth solutions of (1.1) obey the following conservation law :*

$$\partial_t\{\rho E\}(\mathbf{V}^\epsilon)+\nabla.\ \{(\rho E+\sum_{i=1}^N p_i)\mathbf{w}\}(\mathbf{V}^\epsilon)=\varepsilon\ \nabla.\ \big(\sum_{i=1}^N \mu_i(\rho,\rho\epsilon_i)\ \sigma.\mathbf{w}\big)^\epsilon, \tag{1.6}$$

$$\partial_t(\rho S_i)(\mathbf{V}^\epsilon)+\nabla.\ (\rho S_i\mathbf{w})(\mathbf{V}^\epsilon)=-\varepsilon\ \frac{1}{T_i}\mu_i^\epsilon(\rho,\rho\epsilon_i)\ \sigma^\epsilon:\nabla\mathbf{w}^\epsilon,\quad i=1,...,N, \tag{1.7}$$

where the total energy reads $\rho E=||\rho\mathbf{w}||^2/2\rho+\sum_{i=1}^N\rho\epsilon_i$.
Then, it is clear that system (1.1) cannot be recast in full conservation form since generally speaking only one additional conservation law exists, while (1.1) is composed of N equations in non-conservation form. From a numerical point of view, it is well-known that the lack of conservation form makes challenging the numerical approximation of the underlying shock waves.
With this in mind, let us however use the above result to propose a suitable change of variables for (1.1). Obviously, the variables ρ, $\rho\mathbf{w}$ and ρE are natural candidates since they are governed by conservation laws. Next and for completeness, we consider without restriction the set $\{\rho S_i\}_{1\leq i\leq N-1}$ to define $\mathbf{v}:=(\rho,\rho\mathbf{w},\rho E,\{\rho S_i\}_{1\leq i\leq N-1})$ as an admissible change of variables. Then, smooth solutions of (1.1) satisfy :

$$\left\{\begin{array}{l}
\partial_t\rho^\epsilon+\nabla.\ (\rho\mathbf{w})^\epsilon=0,\qquad t>0,\\
\partial_t(\rho\mathbf{w})^\epsilon+\nabla.\ (\rho\mathbf{w}\otimes\mathbf{w}+\sum_{i=1}^N p_i\ \mathbf{I_d})^\epsilon=\varepsilon\ \nabla.\ \big(\sum_{i=1}^N\mu_i(\rho,\rho\epsilon_i)\ \sigma\big)^\epsilon,\\
\partial_t(\rho E)^\epsilon+\nabla.\ \{(\rho E+\sum_{i=1}^N p_i)\mathbf{w}\}^\epsilon=\varepsilon\ \nabla.\ \big(\sum_{i=1}^N\mu_i(\rho,\rho\epsilon_i)\ \sigma.\mathbf{w}\big)^\epsilon,\\
\partial_t(\rho S_i)^\epsilon+\nabla.\ (\rho S_i\mathbf{w})^\epsilon=-\varepsilon\ \frac{1}{T_i}\mu_i^\epsilon(\rho,\rho\epsilon_i)\,\sigma^\epsilon:\nabla\mathbf{w}^\epsilon,\ i=1,...,N-1.
\end{array}\right. \tag{1.8}$$

This equivalent form of (1.1) will be useful in the forthcoming developments. Note that the smooth solutions of (1.8) obey the additional balance law

$$\partial_t\rho S_N(\mathbf{v}^\epsilon)+\nabla.\ (\rho S_N(\mathbf{v})\mathbf{w})^\epsilon=-\varepsilon\ \frac{1}{T_N}\mu_N^\epsilon(\rho,\rho\epsilon_N)\ \sigma^\epsilon:\nabla\mathbf{w}^\epsilon. \tag{1.9}$$

1.2. The asymptotic regime $\varepsilon\ \to 0^+$. In this work, we are concerned with very large values of the Reynolds number, that is very small values of $\varepsilon\ >0$. In this

regime, the solutions of the PDE model (1.8) exhibit very stiff zones of transition, the so-called viscous shock layers and boundary layers. We shall only focus on the viscous shock profiles and postpone the treatment of boundary layers to a forthcoming work. The proper resolution of the small scales taking place within the shock profiles requires grid refinements that are out of reach till now. This means that the small scales cannot be resolved in practice. We are thus led to consider the system (1.8) in the limit of $\varepsilon \to 0$. However, the non-conservation form of the viscous perturbation entering (1.8) makes not straightforward the way to handle this singular limit. Indeed, the entropy dissipation rates in (1.8) cannot vanish in the limit $\varepsilon \to 0$, unlike the viscous terms in conservation form. In other words, the limit system formally writes

$$\partial_t \mathbf{v} + \nabla.\ \mathbf{F}(\mathbf{v}) = \kappa_{\mathbf{v}} \tag{1.10}$$

where $\kappa_{\mathbf{v}} = (0_{\mathbb{R}^{d+2}}, (\kappa_{i\mathbf{v}})_{1\le i\le N-1})$ denotes a vector-valued bounded Borel measure to be precised. This measure is concentrated on the shock discontinuities of the limit solutions of (1.8) and vanishes elsewhere. In particular, $\kappa_{i\mathbf{v}} \le 0$ with $i \in \{1, ..., N-1\}$ stands for the dissipation rate of the entropy S_i across shock discontinuities. In the limit $\varepsilon \to 0$, the additional equation (1.9) reads

$$\partial_t \rho S_N(\mathbf{v}) + \nabla.\ \rho S_N(\mathbf{v})\mathbf{w} = \kappa_{N\mathbf{v}}. \tag{1.11}$$

Let us first give a clear mathematical definition of the discontinuous plane waves of the limit equations (1.10)-(1.11). These solutions have the special form

$$\mathbf{v}(\mathbf{x}, t) \equiv \mathbf{v}(\mathbf{x}.\nu - \sigma t) = \mathbf{v}_- + (\mathbf{v}_+ - \mathbf{v}_-)\ H(\mathbf{x}.\nu - \sigma t), \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.12}$$

where $\mathbf{v}_-$ and $\mathbf{v}_+$ are two constant states, $\sigma \in \mathbb{R}$ denotes the speed of propagation, $\nu \in \mathbb{R}^d$ with $||\nu|| = 1$ is the normal direction of the plane wave and H is the Heavyside function. Arguing about the rotational invariance of the equations, it is classical to shift from the (arbitrary) direction ν to a fixed given direction, say e_1, in order to characterize the shock solutions in a single space variable $x \in \mathbb{R}$:

$$\mathbf{v}^\nu(x - \sigma t) = \mathbf{v}^\nu_- + (\mathbf{v}^\nu_+ - \mathbf{v}^\nu_-)\ H(x - \sigma t), \tag{1.13}$$

where $\mathbf{v}^\nu_+ \equiv \mathbf{v}(\mathbf{v}_+, \nu)$ (respectively $\mathbf{v}^\nu_- \equiv \mathbf{v}(\mathbf{v}_-, \nu)$) is built from $\mathbf{v}_+$ (respectively $\mathbf{v}_-$) and the directions ν and e_1. We refer to [27] for the details. As usual, a function (1.13) is said to be a shock solution if it admits a viscous shock profile. In other words, such a function must be the limit when $\epsilon \to 0$ of a family of smooth traveling wave solutions of the quasi-1D form of the equations (1.8)-(1.9). The latter are solutions of the form $\mathbf{v}(\xi)$, $\xi = x - \sigma t$, with $\lim_{\xi\to\pm\infty} \mathbf{v}(\xi) = \mathbf{v}^\nu_\pm$. Existence and uniqueness (up to translation) of such solutions have been proved in [15], under the usual Lax condition. Namely and focusing on the first characteristic field, being given any $\mathbf{v}^\nu_-$ and velocity σ prescribed such that $\sigma < \mathbf{w}_-.\nu - c(\mathbf{v}^\nu_-)$, existence and uniqueness is proved for $\mathbf{v}^\nu_+$. Integrating the above equations for $\xi \in \mathbb{R}$ yields the expected generalized jump relations

$$-\sigma((\rho S_i)_+ - (\rho S_i)_-) + ((\rho S_i \mathbf{w}.\nu)_+ - (\rho S_i \mathbf{w}.\nu)_-) = \kappa_i(\mathbf{v}^\nu_-, \sigma) \equiv \kappa_i(\mathbf{v}(\mathbf{v}_-, \nu); \sigma)$$

where we have set

$$\kappa_i(\mathbf{v}(\mathbf{v}_-, \nu); \sigma) = -\int_{\mathbb{R}} \frac{\mu_i}{T_i} ||d_\xi \mathbf{w}.\nu||^2\ d\xi. \tag{1.14}$$

At last, in order to give a clear mathematical framework for the limit equations (1.10) and (1.11), we restrict ourselves to piecewise Lipschitz functions with bounded discontinuities. The weak solutions of the limit equations (1.10)-(1.11) are defined as follows. Let $\mathbf{n} = (n_t, \mathbf{n}_x)$ with $\mathbf{n}_x = (n_1, ..., n_d) \in \mathbb{R}^d$ denote an unit outward vector to a time-space smooth discontinuity surface Γ. We write $\mathbf{v}_-$ and $\mathbf{v}_+$ the limit of the function $\mathbf{v}$ at $(t, \mathbf{x}) \in \Gamma$ in the normal direction $\mathbf{n}$:

$$\mathbf{v}_\pm = \lim_{\theta \to 0^\pm} \mathbf{v}(t + \theta n_t, \mathbf{x} + \theta\ \mathbf{n}_x). \tag{1.15}$$

DEFINITION 1.3. *Let $\kappa_{\mathbf{v}}$ be a vector-valued bounded Borel measure. Then, a piecewise Lipschitz function $\mathbf{v}$ is a solution of (1.10) if :*
1) In the zone of smoothness, $\mathbf{v}$ solves in the classical sense

$$\partial_t \mathbf{v} + \nabla .\ \mathbf{F}(\mathbf{v}) = 0. \tag{1.16}$$

2) At a point of discontinuity between $\mathbf{v}_-$ and $\mathbf{v}_+$, $\mathbf{v}$ satisfies the generalized jump conditions :

$$n_t(\mathbf{v}_+ - \mathbf{v}_-) + \sum_{i=1}^{d} n_i(\mathbf{F}_i(\mathbf{v}_+) - \mathbf{F}_i(\mathbf{v}_-)) = \kappa(\mathbf{v}(\mathbf{v}_-, \nu), \sigma), \tag{1.17}$$

where $\kappa(\mathbf{v}(\mathbf{v}_-, \nu), \sigma) = (0, (\kappa_i(\mathbf{v}(\mathbf{v}_-, \nu), \sigma))_{1 \leq i \leq N-1})$ is obtained from (1.14) with $i = 1, ..., N-1$, setting as usual $\nu = \mathbf{n}_x / ||\mathbf{n}_x||_{\mathbb{R}^d}$, $\sigma = -n_t / ||\mathbf{n}_x||_{\mathbb{R}^d}$.

2. Numerical motivation. The objective of this section is to explain from a numerical point of view the motivation of the present work, namely the devise of a robust and implicit in time numerical scheme for approximating the solutions of (1.10)-(1.11). We begin by showing that classical methods like Godunov's scheme fail in providing numerical solutions in good agreement with exact ones. We proceed by a brief review of our previous works [4], [12] and [13] in which several relevant *explicit in time* numerical strategies are proposed. At last, we mention the main difficulties related to the extension of these strategies to an *implicit in time* framework, and give the basis of the developments proposed in the next section.

2.1. What is wrong with the classical methods. In this paragraph, we give example of the failure of the classical finite volume methods for approximating the solutions of (1.10)-(1.11). We focus ourselves on the celebrated Godunov method in one space dimension and with constant space and time steps Δx and Δt. Let us consider the computation of an isolated shock wave $(\mathbf{v}_-, \mathbf{v}_+, \sigma)$ separating two constant states $\mathbf{v}_-$ and $\mathbf{v}_+$ and propagating with speed $\sigma > 0$ such that :

$$-\sigma(\mathbf{v}_+ - \mathbf{v}_-) + (f(\mathbf{v}_+) - f(\mathbf{v}_-)) = \kappa(\mathbf{v}_-, \sigma), \tag{2.1}$$

and

$$-\sigma\big((\rho S_N)(\mathbf{v}_+) - (\rho S_N)(\mathbf{v}_-)\big) + \big((\rho S_N w)(\mathbf{v}_+) - (\rho S_N w)(\mathbf{v}_-)\big) = \kappa_N(\mathbf{v}_-, \sigma). \tag{2.2}$$

We start from the initial data $\mathbf{v}_0(x) = \mathbf{v}_- + (\mathbf{v}_+ - \mathbf{v}_-)H(x)$ where H denotes the Heavyside function. After one time step, the Godunov scheme gives a piecewise constant solution which differs from $\mathbf{v}_-$ and $\mathbf{v}_+$ only in the cell $]0, \Delta x[$ with a state $\overline{\mathbf{v}}_1$ given by :

$$\overline{\mathbf{v}}_1 = \frac{1}{\Delta x} \int_0^{\Delta x} \mathbf{v}_0(x - \sigma \Delta t) dx = \frac{\Delta t}{\Delta x} \sigma \mathbf{v}_- + (1 - \frac{\Delta t}{\Delta x} \sigma) \mathbf{v}_+, \tag{2.3}$$

that is, setting $\lambda = \Delta t/\Delta x$ and using (2.1),

$$\overline{\mathbf{v}}_1 = \mathbf{v}_+ - \lambda(f(\mathbf{v}_+) - f(\mathbf{v}_-)) + \lambda\kappa(\mathbf{v}_-, \sigma). \tag{2.4}$$

This new value $\overline{\mathbf{v}}_1 \neq \mathbf{v}_-, \mathbf{v}_+$ illustrates the numerical diffusion induced by the averaging procedure of the Godunov method. As it is well-known, this numerical diffusion is associated with an entropy dissipation rate that can be measured thanks to the additional entropy law (1.11). Indeed, Jensen's inequality gives :

$$(\rho S_N)(\overline{\mathbf{v}}_1) \leq \frac{1}{\Delta x}\int_0^{\Delta x} (\rho S_N)(\mathbf{v}_0(x-\sigma\Delta t))dx = \lambda\sigma(\rho S_N)(\mathbf{v}_-) + (1-\lambda\sigma)(\rho S_N)(\mathbf{v}_+),$$

that is using (2.2),

$$(\rho S_N)(\overline{\mathbf{v}}_1) \leq (\rho S_N)(\mathbf{v}_+) - \lambda\big((\rho S_N w)(\mathbf{v}_+) - (\rho S_N w)(\mathbf{v}_-)\big) + \lambda\kappa_N(\mathbf{v}_-, \sigma) \tag{2.5}$$

where

$$(\rho S_N)(\overline{\mathbf{v}}_1) - \frac{1}{\Delta x}\int_0^{\Delta x} (\rho S_N)(\mathbf{v}_0(x-\sigma\Delta t)) = \mathcal{O}(1)\frac{1}{\Delta x}\int_0^{\Delta x} ||\overline{\mathbf{v}}_1 - \mathbf{v}_0(x-\sigma\Delta t)||^2 dx$$

for some $\mathcal{O}(1)$ linked to the modulus of convexity of the function $\mathbf{v} \to (\rho S_N)(\mathbf{v})$.
Hence, the error between $(\rho S_N)(\overline{\mathbf{v}}_1)$ and the expected value $\lambda\sigma(\rho S_N)_- + (1-\lambda\sigma)(\rho S_N)_+$ evolves according to the square of the shock strength. This explains the unacceptable errors generally observed between the exact and the numerical solutions after several numerical time steps. We refer the reader to [4], [12], [13], [14] for several illustrations.
Note that by (2.4) we have

$$\overline{\mathbf{v}}_1 - \mathbf{v}_+ + \lambda(f(\mathbf{v}_+) - f(\mathbf{v}_-)) \leq$$

$$\Big((\rho S_N)(\overline{\mathbf{v}}_1) - (\rho S_N)(\mathbf{v}_+) + \lambda\big((\rho S_N w)(\mathbf{v}_+) - (\rho S_N w)(\mathbf{v}_-)\big)\Big)\frac{\kappa(\mathbf{v}_-, \sigma)}{\kappa_N(\mathbf{v}_-, \sigma)},$$

whereas an equality would be hoped.

2.2. How to correct the classical methods. In order to get a good agreement between exact and numerical solutions, we suggest modifying the classical Godunov scheme. The correction we propose intends to keep the discrete dissipation rates in balance with the kinetic functions. In other words, and still for the computation of an isolated shock wave $(\mathbf{v}_-, \mathbf{v}_+, \sigma)$, we propose to substitute $\overline{\mathbf{v}}_1$ for a constant value $\mathbf{v}_1^\sharp$ solution to the following discrete equations :

$$\mathbf{v}_1^\sharp - \mathbf{v}_+ + \lambda(f(\mathbf{v}_+) - f(\mathbf{v}_-)) = \tag{2.6}$$

$$\Big(\ (\rho S_N)(\mathbf{v}_1^\sharp) - (\rho S_N)(\mathbf{v}_+) + \lambda\big((\rho S_N w)(\mathbf{v}_+) - (\rho S_N w)(\mathbf{v}_-)\big)\ \Big)\frac{\kappa(\mathbf{v}_-, \sigma)}{\kappa_N(\mathbf{v}_-, \sigma)}.$$

The extension of this procedure to general initial data is immediate and writes :

$$\mathbf{v}_j^{n+1} - \mathbf{v}_j^n + \lambda(f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}) = \tag{2.7}$$

$$\Big(\ (\rho S_N)(\mathbf{v}_j^{n+1}) - (\rho S_N)(\mathbf{v}_j^n) + \lambda((\rho S_N w)^n_{j+\frac{1}{2}} - (\rho S_N w)^n_{j-\frac{1}{2}})\ \Big)\frac{< \kappa, \mathcal{C}_j^n >}{< \kappa_N, \mathcal{C}_j^n >}$$

where $f^n_{j+\frac{1}{2}}$ and $(\rho S_N \mathbf{w})^n_{j+\frac{1}{2}}$ denote the usual Godunov flux and entropy flux functions, while $< \kappa, \mathcal{C}_j^n >$ and $< \kappa_N, \mathcal{C}_j^n >$ denote the mass of the kinetic functions summed over all the shock waves propagating in the cell $\mathcal{C}_j^n$.
Existence and uniqueness of a solution to (2.7) is proved in [4], [12].

2.3. Existing time-explicit correction procedures. In this paragraph, we briefly review the works [4], [12], and [13] in order to highlight that the exact value of the ratio of the kinetic functions involved in (2.7) can be fairly well approximated. Actually, several robust approximations exist in a time explicit setting. Their review will motivate the introduction of a new correction procedure devoted to large time stepping methods.

The first approximation naturally arises from the original equations (1.8) for a given $\varepsilon > 0$. We have the following identities:

$$\frac{T_i^\epsilon}{\mu_i^\epsilon}\big(\ \partial_t(\rho S_i)^\epsilon + \nabla.\ (\rho S_i \mathbf{w})^\epsilon\ \big) = \frac{T_N^\epsilon}{\mu_N^\epsilon}\big(\ \partial_t(\rho S_N)(\mathbf{v}^\epsilon) + \nabla.\ (\rho S_N \mathbf{w})(\mathbf{v}^\epsilon)\ \big) \tag{2.8}$$

which formally yields in the limit $\epsilon \to 0$ and for all $i = 1, \dots, N-1$:

$$\partial_t(\rho S_i) + \nabla.\ (\rho S_i \mathbf{w}) = \widetilde{\frac{\mu_i T_N}{\mu_N T_i}}\big(\ \partial_t(\rho S_N)(\mathbf{v}) + \nabla.\ (\rho S_N \mathbf{w})(\mathbf{v})\ \big). \tag{2.9}$$

In the smooth zones of the exact solution, the above relations reduce to

$$\partial_t(\rho S_i) + \nabla.\ (\rho S_i \mathbf{w}) = 0, \tag{2.10}$$

while at a point of discontinuity between $\mathbf{v}_-$ and $\mathbf{v}_+$, they coincide with the following generalized jump relations :

$$n_t\big((\rho S_i)_+ - (\rho S_i)_-\big) + \sum_{k=1}^{d} n_k\big((\rho S_i \mathbf{w})_+ - (\rho S_i \mathbf{w})_-\big) = \tag{2.11}$$

$$\frac{\kappa_i(\mathbf{v}_-, \sigma)}{\kappa_N(\mathbf{v}_-, \sigma)}\Big(\ n_t\big((\rho S_N)(\mathbf{v}_+) - (\rho S_N)(\mathbf{v}_-)\big) + \sum_{k=1}^{d} n_k\big((\rho S_N \mathbf{w})(\mathbf{v}_+) - (\rho S_N \mathbf{w})(\mathbf{v}_-)\big)\ \Big),$$

provided that the unspecified value of $\widetilde{\frac{\mu_i T_N}{\mu_N T_i}}$ is accordingly defined. Extensive calculations proposed in [11] and [12] strongly support that the discrete solutions stay largely free from the local definition of the ratio $\frac{\mu_i T_N}{\mu_N T_i}$. Thus, the latter provides a robust approximation of the ratios of kinetic functions in (2.7).
Let us underline that the resulting method is nonlinear in each cell due to the nonlinearity in $\mathbf{v} \to (\rho S_N)(\mathbf{v})$. The well-posedness and stability properties of this nonlinear method are investigated in [4], [12].

In [13], a variant is derived. This variant makes use of the second principle in order to recast (2.9) when considering internal energies. Easy calculations lead for $i = 1, \dots, N-1$ to :

$$\partial_t \rho\epsilon_i + \nabla.\ (\rho\epsilon_i\ \mathbf{w}) + p_i \nabla.\ \mathbf{w} = \widetilde{\frac{\mu_i}{\mu_N}}\big(\partial_t \rho\epsilon_N + \nabla.\ (\rho\epsilon_N\ \mathbf{w}) + p_N(\mathbf{v})\nabla.\ \mathbf{w}\big), \tag{2.12}$$

and by summation, introducing $\rho\epsilon = \sum_i \epsilon_i,\ \mathcal{P}(\mathbf{v}) = \sum_i p_i(\mathbf{v})$, to :

$$\partial_t \rho\epsilon_i + \nabla.\ (\rho\epsilon_i\ \mathbf{w}) + p_i \nabla.\ \mathbf{w} = \frac{\widetilde{\frac{\mu_i}{\mu_N}}}{\sum_{j=1}^{N} \widetilde{\frac{\mu_j}{\mu_N}}}\big(\partial_t \rho\epsilon + \nabla.\ (\rho\epsilon\ \mathbf{w}) + \mathcal{P}(\mathbf{v})\nabla.\ \mathbf{w}\big). \tag{2.13}$$

In [13], we show how to take advantage of (2.13) and the linearity of the total internal energy with respect to the internal energies $(\rho\epsilon_i)_{i=1,\dots,N}$ to derive *linear* approximate solvers extending (2.9) to (2.13) and which preserve the positiveness of all the $(\rho\epsilon)_i$, *in a time-explicit setting*.
We underline that in this method, the time derivative of the total internal energy $\rho\epsilon$ is known from the evolution in time of the density, momentum and total energy thanks to the relation $\rho\epsilon = \rho E - ||\rho\mathbf{w}||^2/2\rho$.

2.4. Towards a time-implicit correction procedure. The extension of the latter procedure to a time-implicit setting is difficult and for this reason, we will use in the core of the present work a new version of (2.13).
In the first hand, proposing a stable time integration of the internal energy laws (2.13) with large time stepping is not obvious. With this in mind and invoking again the second principle of thermodynamics, we first propose to replace the left hand side of (2.13) by transport equations :

$$\partial_t S_i + \mathbf{w}.\nabla S_i = \frac{\widetilde{\frac{\mu_i T_i}{\mu_N}}}{\sum_{j=1}^N \widetilde{\frac{\mu_j}{\mu_N}}}\big(\partial_t\epsilon + \mathbf{w}.\nabla\epsilon + \frac{\mathcal{P}}{\rho}\nabla.\ \mathbf{w}\big), \quad i = 1,\dots,N-1. \tag{2.14}$$

On the second hand, forcing the positiveness of the specific entropies $(S_i)_{i=1,\dots,N-1}$ and above all of S_N which is obtained by the relation

$$\rho E - \frac{||\rho\mathbf{w}||^2}{2\rho} = \rho\epsilon = \sum_{j=1}^{N-1}\rho\epsilon_j(\rho, S_j) + \rho\epsilon_N(\rho, S_N),$$

is pretty challenging. To immediately force the positiveness of the $(N-1)$ ratios $\frac{S_i}{S_N}$, we suggest

$$\partial_t X_i + \mathbf{w}.\nabla X_i = C_i\big(\partial_t\rho\epsilon + \nabla.\ (\rho\epsilon\ \mathbf{w}) + \mathcal{P}\nabla.\ \mathbf{w}\big)\ \ i = 1,\dots,N-1, \tag{2.15}$$

$$\text{with}\quad X_i = Ln\frac{S_i}{S_N}\quad\text{and}\quad C_i = \frac{1}{\rho\sum_{j=1}^N\mu_j}\Big(\frac{\mu_N}{T_N S_N} - \frac{\mu_i}{T_i S_i}\Big).$$

Then, the N specific entropies are recovered from the knowledge of the $(N-1)$ $(X_i)_{1\leq 1\leq N-1}$ by $S_i = S_N exp(X_i)$ for $i = 1, ..., N-1$, and solving in S_N :

$$\rho\epsilon = \sum_{j=1}^{N-1}\rho\epsilon_j(\rho, S_N exp(X_j)) + \rho\epsilon_N(\rho, S_N). \tag{2.16}$$

We shall prove hereafter that from a numerical standpoint, such a formulation is relevant to get a positive S_N to this equation and thus positive values for all the other $(N-1)$ entropies S_i.

3. Numerical approximation. In this section, we address the numerical approximation of the solutions of (1.1). The proposed method can be understood as a predictor-corrector strategy. In particular, one of the objectives of the correction step will be to impose the validity of the compatibility relations (2.15) at the discrete level.
The outline of this section is as follows. We begin by some basics and notations (section 3.1). Then we describe our algorithm first using an explicit time integration (section 3.2). At last, a time-implicit numerical scheme of the multi-pressure gas dynamics equations is proposed in section 3.3.

3.1. Basics and notations. Let $t^n = n\Delta t$ be a uniform discretization of $\mathbb{R}_+$ and, for each $h > 0$, let $\mathcal{T}_h$ be a triangulation of $\mathbb{R}^d$ made of nonoverlapping and nonempty polyhedra. As is customary in the finite volume framework, we assume that if K and K' in $\mathcal{T}_h$ have a nonempty intersection, say I, then I is either a face of both K and K' or I has a Hausdorff dimension less than $d-1$. The set of the faces of a polyhedron K is denoted by ∂K and for each face on K, $\mathbf{n}_{e,K} \in \mathbb{R}^d$ represents the outward unit normal to the face e. Given a face e of K, K_e is the unique polyhedron in $\mathcal{T}_h$ which shares the same face e with K. The volume of K and the $d-1$ measure of e are denoted by $|K|$ and $|e|$ respectively. Without loss of generality, we assume that $h = \sup_{K\in\mathcal{T}_h} h_K < +\infty$ where h_K is the exterior diameter of the polyhedron K. The perimeter of K is defined by $p_K = \sum_{e\in\partial K} |e|$.
The approximate solutions of (1.1) are sought under the form of a piecewise constant function $\mathbf{v}_h$

$$\mathbf{v}_h(t,\mathbf{x}) = \mathbf{v}_K^n \qquad (t,\mathbf{x}) \in [t^n, t^{n+1}[\times K. \tag{3.1}$$

3.2. Explicit in time numerical scheme. As already mentioned, the whole strategy is of predictor-corrector kind. Before entering the details, we begin by a rough description of the two steps.
In the prediction step we propose to solve approximately in the slab $[0, \Delta t]$, Δt small enough, the following system :

$$\left\{\begin{array}{l} \partial_t \rho + \nabla.\ \rho\mathbf{w} = 0, \qquad t > 0, \\ \partial_t \rho\mathbf{w} + \nabla.\ (\rho\mathbf{w}\otimes\mathbf{w} + \mathcal{P}(\mathbf{u})\ \mathbf{I_d}) = 0, \\ \partial_t \rho S_i + \nabla.\ \rho S_i\mathbf{w} = 0, \quad i = 1, ..., N. \end{array}\right. \tag{3.2}$$

To shorten the notations, we introduce $\mathbf{u} = (\rho, \rho\mathbf{w}, (\rho S_i)_{i=1,...,N})$. The weak solutions of system (3.2) are selected by the entropy-like inequality :

$$\partial_t(\rho E)(\mathbf{u}) + \nabla.\ (\rho\mathcal{H}\mathbf{w})(\mathbf{u}) \leq 0, \quad (\rho\mathcal{H}\mathbf{w})(\mathbf{u}) = \big((\rho E)(\mathbf{u}) + \mathcal{P}(\mathbf{u})\big)\mathbf{w}. \tag{3.3}$$

We notice that system (3.2) considers the last entropy ρS_N as a main unknown. Meanwhile, the total energy ρE becomes a function of $\mathbf{u} = \big(\rho, \rho\mathbf{w}, (\rho S_i)_{1\leq i\leq N}\big)^t$ and (3.3) an additional evolution law acting as a selection principle. Let us indeed observe that the thermodynamic assumptions made in the introduction actually ensures that the mapping $\mathbf{u} \to (\rho E)(\mathbf{u})$ is strictly convex and may then serve as a natural Lax entropy for (3.2). We refer the reader to [13] for more details.
It will be useful hereafter to notice that each specific entropy S_i of (3.3) satisfies the following transport equation :

$$\partial_t S_i + \mathbf{w}.\nabla S_i = 0, \quad \mathcal{D}' \tag{3.4}$$

for each piecewise Lipschitz continuous weak solutions of (3.2)-(3.3), see [13].
System (3.2) is given the following short form with clear definition for $\mathbf{G}$:

$$\partial_t \mathbf{u} + \nabla.\ \mathbf{G}(\mathbf{u}) = 0. \tag{3.5}$$

Then, a correction step is compulsory. It aims at simultaneously restoring the conservation of the total energy ρE

$$\partial_t \rho E + \nabla.\ \big(\rho E + \mathcal{P}(u)\big)\mathbf{w} = 0, \tag{3.6}$$

and imposing the validity of the compatibility relations

$$\partial_t X_i + \mathbf{w}.\nabla X_i = C_i(\partial_t \epsilon + \mathbf{w}.\nabla \epsilon + \frac{\mathcal{P}(\mathbf{u})}{\rho}\nabla.\ \mathbf{w}), \quad i = 1, ..., N-1. \tag{3.7}$$

Let us now go into details.

Prediction step ($t^n \rightarrow t^{n+1-}$). Within a fully time-explicit setting, the states $\mathbf{u}_K^n$ are advanced in time according to a finite volume approximation of (3.2) which general form is

$$\mathbf{u}_K^{n+1-} = \mathbf{u}_K^n - \frac{\Delta t}{|K|}\sum_{e\in\partial K} \mathcal{G}_{e,K}^n |e|, \tag{3.8}$$

where the numerical flux $\mathcal{G}_{e,K}^n = \mathcal{G}(\mathbf{u}_K^n, \mathbf{u}_{K_e}^n; \mathbf{n}_{e,K})$ is built from a two-point Lipschitz-continuous flux function $\mathcal{G}$ at each face e of each polyhedron K. This flux function is expected to satisfy the following two properties :

$$\begin{cases} \mathcal{G}(\mathbf{u}_K, \mathbf{u}_{K_e}; \mathbf{n}_{e,K}) + \mathcal{G}(\mathbf{u}_{K_e}, \mathbf{u}_K; \mathbf{n}_{e,K_e}) = 0, & \text{(conservation)} \\ \\ \forall\ \mathbf{u} \in \Omega, \quad \mathbf{u} = \mathbf{u}_K = \mathbf{u}_{K_e} \implies \mathcal{G}(\mathbf{u}, \mathbf{u}; \mathbf{n}_{e,K}) = \mathbf{G}(\mathbf{u}).\mathbf{n}_{e,K}. & \text{(consistency)} \end{cases} \tag{3.9}$$

Arguing about the rotational invariance of the equations, the definition of $\mathcal{G}$ classically follows from the following quasi-1D form of (3.2) :

$$\partial_t \mathbf{u} + \partial_x \mathbf{G}_{i_1}(\mathbf{u}) = 0, \tag{3.10}$$

with the exact flux function $\mathbf{G}_{i_1}$ given by

$$\mathbf{G}_{i_1} = \big(\rho w_1, \rho w_1^2 + \mathcal{P}(\mathbf{u}), (\rho w_1 w_i)_{2\le i\le d}, (\rho S_j w_1)_{1\le j\le N}\big) \tag{3.11}$$

(here i_1 denotes the first unit vector of the canonical basis of $\mathbb{R}^d$), and supplemented with the validity of the entropy inequality

$$\partial_t(\rho E)(\mathbf{u}) + \partial_x(\rho \mathcal{H} w_1)(\mathbf{u}) \le 0, \quad (\rho\mathcal{H}w_1)(\mathbf{u}) = ((\rho E)(\mathbf{u}) + \mathcal{P}(\mathbf{u}))w_1. \tag{3.12}$$

More precisely, given a two-point numerical flux function $\mathcal{G}(.,.;i_1) : \Omega_{\mathbf{u}} \times \Omega_{\mathbf{u}} \rightarrow \mathbb{R}^{d+N+1}$ consistent with $\mathbf{G}_{i_1}$, one sets

$$\mathcal{G}_{e,K}^n = \mathcal{G}(\mathbf{u}_K^n, \mathbf{u}_{K_e}^n; \mathbf{n}_{e,K}) = T_{e,K}^{-1}\ \mathcal{G}(T_{e,K}\mathbf{u}_K^n, T_{e,K}\mathbf{u}_{K_e}^n; i_1) \tag{3.13}$$

where $T_{e,K}$ is classically built from a rotation operator in $\mathbb{R}^d$ mapping i_1 to $\mathbf{n}_{e,K}$. Let us just recall that $T_{e,K}$ only acts on the velocity $\mathbf{w}$ and keeps unchanged all the other components of $\mathbf{u}$, with the property that $(\rho E)(T_{e,K}\mathbf{u}) = (\rho E)(\mathbf{u})$ for all $\mathbf{u}$ (the kinetic energy $||\rho\mathbf{w}||^2/2\rho$ is the same). By definition, $T_{e,K}$ verifies

$$\mathbf{G}(\mathbf{u}).\mathbf{n}_{e,K} = T_{e,K}^{-1}\mathbf{G}_{i_1}(T_{e,K}\mathbf{u}) \tag{3.14}$$

for any given constant state $\mathbf{u} \in \Omega_{\mathbf{u}}$ so that the consistency property in (3.9) is met as soon as $\mathcal{G}(\mathbf{u}, \mathbf{u}; i_1) = \mathbf{G}_{i_1}(\mathbf{u})$. The latter and the conservation property in (3.9) will be satisfied by the definition of $\mathcal{G}(.,.;i_1)$ that we give right now.
For simplicity in the notations and without any restriction (see remark below), we propose to use the Godunov numerical flux function associated with (3.10)-(3.12)

in order to define $\mathcal{G}(.,.;i_1)$. More precisely, denoting $\mathcal{W}(\frac{x}{t};\mathbf{u}_L,\mathbf{u}_R)$ the self-similar solution of the Cauchy problem for (3.10)-(3.12) with initial data

$$\mathbf{u}_0(x)=\mathbf{u}_L \text{ if } x<0,\ \mathbf{u}_R \text{ if } x>0, \tag{3.15}$$

the numerical flux function $\mathcal{G}(.,.;i_1)$ then reads :

$$\mathcal{G}(\mathbf{u}_L,\mathbf{u}_R;i_1)=\mathbf{G}_{i_1}\big(\mathcal{W}(0^+;\mathbf{u}_L,\mathbf{u}_R)\big). \tag{3.16}$$

Here $\mathbf{u}_L$ and $\mathbf{u}_R$ are two constant states in Ω. Note that in the following, we always assume that the $\mathbf{u}_L$ and $\mathbf{u}_R$ never give birth to vacuum in $\mathcal{W}(.;\mathbf{u}_L,\mathbf{u}_R)$ so that the density ρ stays strictly positive hereafter.
It is then clear that $\mathcal{G}(\mathbf{u},\mathbf{u};i_1)=\mathbf{G}_{i_1}(\mathbf{u})$ whereas as far as the conservation property in (3.9) is concerned we have

$$\mathcal{G}(\mathbf{u}_K,\mathbf{u}_{K_e};\mathbf{n}_{e,K})+\mathcal{G}(\mathbf{u}_{K_e},\mathbf{u}_K;\mathbf{n}_{e,K_e})=$$

$$T_{e,K}^{-1}\ \mathcal{G}(T_{e,K}\mathbf{u}_K,T_{e,K}\mathbf{u}_{K_e};i_1)+T_{e,K_e}^{-1}\ \mathcal{G}(T_{e,K_e}\mathbf{u}_{K_e},T_{e,K_e}\mathbf{u}_K;i_1)=$$

$$T_{e,K}^{-1}\ \mathbf{G}_{i_1}\big(\mathcal{W}(0^+;T_{e,K}\mathbf{u}_K,T_{e,K}\mathbf{u}_{K_e})\big)+T_{e,K_e}^{-1}\ \mathbf{G}_{i_1}\big(\mathcal{W}(0^+;T_{e,K_e}\mathbf{u}_{K_e},T_{e,K_e}\mathbf{u}_K)\big).$$

But by the rotational invariance we have $T_{e,K}^{-1}\mathcal{W}(0^+;T_{e,K}\mathbf{u}_K,T_{e,K}\mathbf{u}_{K_e})=$
$T_{e,K_e}^{-1}\mathcal{W}(0^+;T_{e,K_e}\mathbf{u}_{K_e},T_{e,K_e}\mathbf{u}_K)$ so that by (3.14)

$$\mathcal{G}(\mathbf{u}_K,\mathbf{u}_{K_e};\mathbf{n}_{e,K})+\mathcal{G}(\mathbf{u}_{K_e},\mathbf{u}_K;\mathbf{n}_{e,K_e})=$$

$$\mathbf{G}\big(T_{e,K}^{-1}\mathcal{W}(0^+;T_{e,K}\mathbf{u}_K,T_{e,K}\mathbf{u}_{K_e})\big).\mathbf{n}_{e,K}+\mathbf{G}\big(T_{e,K}^{-1}\mathcal{W}(0^+;T_{e,K}\mathbf{u}_K,T_{e,K}\mathbf{u}_{K_e})\big).\mathbf{n}_{e,K_e}.$$

This quantity obviously equals 0 since $\mathbf{n}_{e,K}+\mathbf{n}_{e,K_e}=0$.

The main stability estimates associated with this prediction step are gathered in the next statement, the proof being detailed in [18].

PROPOSITION 3.1. *Under the CFL condition*

$$\max_{\mathbf{u}}\max_{K\in\mathcal{T}_h}\Big\{\frac{\Delta t\ p_K}{|K|}\max_{e\in\partial K}(|\mathbf{w}.\mathbf{n}_{e,K}-c(\mathbf{u})|,|\mathbf{w}.\mathbf{n}_{e,K}+c(\mathbf{u})|)\Big\}\leq\frac{1}{2}, \tag{3.17}$$

for all the $\mathbf{u}$ *under consideration, the finite volume method (3.8)-(3.13)-(3.16) obeys the following discrete entropy like inequality :*

$$(\rho E)(\mathbf{u}_K^{n+1-})-(\rho E)(\mathbf{u}_K^n)+\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\rho\mathcal{H}\mathbf{w})_{e,K}^n\leq 0 \tag{3.18}$$

where with clear notations

$$(\rho\mathcal{H}\mathbf{w})_{e,K}^n=\big((\rho E+\mathcal{P})w_1\big)\big(\mathcal{W}(0^+;T_{e,K}\mathbf{u}_K^n,T_{e,K}\mathbf{u}_{K_e}^n)\big). \tag{3.19}$$

In addition, the following maximum principle holds true :

$$\min_{K\in\mathcal{T}_h}(S_i)_K^n\leq(S_i)_K^{n+1-}\leq\max_{K\in\mathcal{T}_h}(S_i)_K^n,\qquad i=1,...,N. \tag{3.20}$$

Remark. From Harten, Lax and van Leer [28], it is well-known that we may use instead of the Godunov flux function (3.16) any suitable approximate Riemann solvers

to preserve the stability properties given in proposition 3.1.

Correction step ($t^{n+1-} \to t^{n+1}$). This step aims at restoring the conservation of the total energy (3.6) and the validity of the generalized jump conditions (3.7). Let us first show how to derive a consistant finite volume formula for the right hand side of (3.7) which we rewrite for convenience as

$$\sum_{i=1}^{N} \big(\partial_t \epsilon_i(\rho, S_i) + \partial_\rho \epsilon_i(\rho, S_i)\ \mathbf{w}.\nabla\rho + \partial_{S_i} \epsilon_i(\rho, S_i)\ \mathbf{w}.\nabla S_i + \frac{\mathcal{P}_i(\mathbf{u})}{\rho} \nabla.\ \mathbf{w}\big). \tag{3.21}$$

This term worths indeed a particular attention since it is written in non-conservation form. The final statement in that direction is the proposition 3.3 below which provide us with a natural time-explicit finite volume approximation for

$$\partial_\rho \epsilon_i(\rho, S_i)\ \mathbf{w}.\nabla\rho + \partial_{S_i} \epsilon_i(\rho, S_i)\ \mathbf{w}.\nabla S_i + \frac{\mathcal{P}_i(\mathbf{u})}{\rho} \nabla.\ \mathbf{w}, \quad i = 1, ..., N. \tag{3.22}$$

To begin with, we state an easy lemma.

LEMMA 3.2. *At each edge e of each cell K, let us decompose the components of Godunov numerical flux function $\mathcal{G}^n_{e,K}$ for the density and for each entropy ρS_i according to :*

$$(\rho w_1)^n_{e,K} = \rho^n_{e,K}\ (w_1)^n_{e,K} \tag{3.23}$$

$$(\rho S_i w_1)^n_{e,K} = (\rho w_1)^n_{e,K}\ (S_i)^n_{e,K} \tag{3.24}$$

where by construction

$$\rho^n_{e,K} = \rho\big(\mathcal{W}(0^+; T_{e,K}\mathbf{u}^n_K, T_{e,K}\mathbf{u}^n_{K_e})\big), \tag{3.25}$$

$$(w_1)^n_{e,K} = w_1\big(\mathcal{W}(0^+; T_{e,K}\mathbf{u}^n_K, T_{e,K}\mathbf{u}^n_{K_e})\big), \tag{3.26}$$

and

$$\begin{aligned}(S_i)^n_{e,K} &= S_i\big(\mathcal{W}(0^+; T_{e,K}\ \mathbf{u}^n_K, T_{e,K}\ \mathbf{u}^n_{K_e})\big) \\ &= \begin{cases} (S_i)^n_K & \text{if} \quad (\rho w_1)^n_{e,K} > 0, \\ (S_i)^n_{K_e} & \text{otherwise.} \end{cases}\end{aligned} \tag{3.27}$$

Then the discrete density ρ^{n+1-}_K updates equivalently according to

$$\rho^{n+1-}_K = \rho^n_K - \frac{\Delta t}{|K|} \sum_{e\in\partial K} (w_1)^n_{e,K}(\rho^n_{e,K} - \rho^n_K)|e| - \frac{\Delta t}{|K|}\rho^n_K \sum_{e\in\partial K} (w_1)^n_{e,K}|e|, \tag{3.28}$$

and each specific entropy S_i verifies

$$(S_i)^{n+1-}_K = (S_i)^n_K - \frac{\Delta t}{|K|} \sum_{e\in\partial K} \frac{(\rho w_1)^n_{e,K}}{\rho^{n+1-}_K}\big((S_i)^n_{e,K} - (S_i)^n_K\big)|e|, \quad i = 1, ..., N. \tag{3.29}$$

Proof. The proposed identities (3.28) and (3.29) readily follow from (3.8) and the definitions (3.23) and (3.24). □

Let us briefly comment the equivalent formula (3.28). It is first clear that

$$\frac{1}{|K|}\rho^n_K \sum_{e\in\partial K} (w_1)^n_{e,K}|e| \tag{3.30}$$

is a relevant finite volume approximation of $\frac{1}{|K|}\int_K \rho\nabla.\ \mathbf{w}\ d\mathcal{K}$. Henceforth, the quantity $\frac{1}{|K|}\sum_{e\in\partial K}(w_1)^n_{e,K}(\rho^n_{e,K}-\rho^n_K)|e|$ may be understood as a consistant finite volume formula of

$$\frac{1}{|K|\Delta t}\int_0^{\Delta t}\int_K \mathbf{w}.\nabla\rho\ d\mathcal{K}\ dt, \tag{3.31}$$

since first $\nabla.\ \rho\mathbf{w}=\rho\nabla.\ \mathbf{w}+\mathbf{w}.\nabla\rho$ and then the time derivative $(\rho_K^{n+1-}-\rho_K^n)/\Delta t$ is by construction a consistant finite volume approximation of $-\frac{1}{|K|\Delta t}\int_0^{\Delta t}\int_K \nabla.\ (\rho\mathbf{w})\ d\mathcal{K}\ dt$. Recall indeed that $\partial_t\rho+\nabla.\ \rho\mathbf{w}=0$.
In the same way, the identity (3.29) expresses that

$$\frac{1}{|K|}\sum_{e\in\partial K}\frac{(\rho w_1)^n_{e,K}}{\rho_K^{n+1-}}\big((S_i)^n_{e,K}-(S_i)^n_K\big)|e| \tag{3.32}$$

may serve as a natural finite volume formula for

$$\frac{1}{|K|\Delta t}\int_0^{\Delta t}\int_K \mathbf{w}.\nabla S_i\ d\mathcal{K}\ dt, \tag{3.33}$$

reflecting that each specific entropy S_i satisfies the transport equation (3.4).
On the ground of formulae (3.30), (3.31) and (3.33), we are now in position to propose a natural finite volume approximation of (3.22).

Proposition 3.3. *Let us define from the prediction step the following finite formula for approximating (3.22)*

$$\begin{aligned}
&\frac{\Delta t}{|K|}\overline{\partial_\rho\epsilon_i(\rho,S_i)}^{n,n+1-}_K\sum_{e\in\partial K}(w_1)^n_{e,K}(\rho^n_{e,K}-\rho^n_K)|e|\\
&+\frac{\Delta t}{|K|}\overline{\partial_{S_i}\epsilon_i(\rho,S_i)}^{n,n+1-}_K\sum_{e\in\partial K}\frac{(\rho w_1)^n_{e,K}}{\rho_K^{n+1-}}\big((S_i)^n_{e,K}-(S_i)^n_K\big)|e|\\
&+\frac{\Delta t}{|K|}\rho^n_K\overline{\partial_\rho\epsilon_i(\rho,S_i)}^{n,n+1-}_K\sum_{e\in\partial K}(w_1)^n_{e,K}|e|\equiv-(\xi_i)^{n,n+1-}_K
\end{aligned} \tag{3.34}$$

where we have set

$$\overline{\partial_\rho\epsilon_i(\rho,S_i)}^{n,n+1-}_K=\frac{\epsilon_i(\rho_K^{n+1-},(S_i)^{n+1-}_K)-\epsilon_i(\rho^n_K,(S_i)^{n+1-}_K)}{\rho_K^{n+1-}-\rho^n_K} \tag{3.35}$$

$$\overline{\partial_{S_i}\epsilon_i(\rho,S_i)}^{n,n+1-}_K=\frac{\epsilon_i(\rho^n_K,(S_i)^{n+1-}_K)-\epsilon_i(\rho^n_K,(S_i)^n_K)}{(S_i)^{n+1-}_K-(S_i)^n_K}. \tag{3.36}$$

Then, the following identity holds true :

$$(\xi_i)^{n,n+1-}_K=(\epsilon_i)^{n+1-}_K-(\epsilon_i)^n_K \tag{3.37}$$

when defining from the prediction step

$$(\epsilon_i)^{n+1-}_K=\epsilon_i(\rho_K^{n+1-},(S_i)^{n+1-}_K). \tag{3.38}$$

Let us stress that $(\partial_\rho \epsilon_i)_{S_i} = \frac{\mathcal{P}_i}{\rho^2}$ *so that the weight* $\rho_K^n \ \overline{\partial_\rho \epsilon_i(\rho, S_i)}_K^{n,n+1-}$ *is indeed consistant with the required factor* $\frac{\mathcal{P}_i}{\rho}$ *in (3.22).*

Proof. Starting from the definitions (3.35), (3.36) and (3.38), we first observe that

$$(\epsilon_i)_K^{n+1-} - (\epsilon_i)_K^n = \overline{\partial_\rho \epsilon_i(\rho, S_i)}_K^{n,n+1-} (\rho_K^{n+1-} - \rho_K^n) + \overline{\partial_{S_i} \epsilon_i(\rho, S_i)}_K^{n,n+1-} \big((S_i)_K^{n+1-} - (S_i)_K^n\big).$$

Then plugging (3.28) and (3.29) yields (3.37) with definition (3.34). □
Equipped with the natural approximation (3.34) of (3.22) (and its equivalent form (3.37)), we now describe precisely the correction step we have briefly sketched in subsection 2.4. First, we propose in each cell K to keep unchanged the updated values of the conservative variables ρ and $\rho\mathbf{w}$:

$$\rho_K^{n+1} = \rho_K^{n+1-}, \quad (\rho\mathbf{w})_K^{n+1} = \quad (\rho\mathbf{w})_K^{n+1-}, \tag{3.39}$$

and to restore the conservation of the total energy when defining from (3.18)

$$(\rho E)_K^{n+1} = (\rho E)_K^n - \frac{\Delta t}{|K|} \sum_{e \in \partial K} (\rho \mathcal{H}\mathbf{w})_{e,K}^n |e|. \tag{3.40}$$

This allows to update the total internal energy thanks to

$$(\rho\epsilon)_K^{n+1} = (\rho E)_K^{n+1} - \frac{||(\rho\mathbf{w})_K^{n+1}||^2}{2\rho_K^{n+1}}. \tag{3.41}$$

Next, in order to enforce the validity of the generalized jump conditions (3.7) at the discrete level, we propose to define $(S_i)_K^{n+1}$ from the solutions $(X_i)_K^{n+1}$ of

$$\begin{aligned}(X_i)_K^{n+1} - (X_i)_K^n + \frac{\Delta t}{|K|} \sum_{e \in \partial K} \frac{(\rho w_1)_{e,K}^n}{\rho_K^{n+1-}} \big((X_i)_{e,K}^n - (X_i)_K^n\big) |e| \\ = C_i(\epsilon_K^{n+1} - \epsilon_K^n + \xi_K^{n,n+1-}), \quad i = 1, ..., N-1,\end{aligned} \tag{3.42}$$

where we have set

$$\xi_K^{n,n+1-} = \sum_{i=1}^N (\xi_i)_K^{n,n+1-} \tag{3.43}$$

and in agreement with (3.33),

$$(X_i)_{e,K}^n = \begin{cases} (X_i)_K^n & \text{if} \quad (\rho w_1)_{e,K}^n > 0, \\ (X_i)_{K_e}^n & \text{otherwise.} \end{cases} \tag{3.44}$$

Observe from (3.43) and the proposition 3.3, that (3.42) actually provides us with a natural finite volume approximation of (3.7). Next, invoking (3.37) for each $i = 1, ..., N$, (3.42) just recasts equivalently as

$$\begin{aligned}(X_i)_K^{n+1} - (X_i)_K^n + \frac{\Delta t}{|K|} \sum_{e \in \partial K} \frac{(\rho w_1)_{e,K}^n}{\rho_K^{n+1-}} \big((X_i)_{e,K}^n - (X_i)_K^n\big) |e| \\ = C_i(\epsilon_K^{n+1} - \epsilon_K^{n+1-}) \quad i = 1, ..., N-1,\end{aligned} \tag{3.45}$$

where by definition

$$\epsilon_K^{n+1-} = \sum_{i=1}^{N} (\epsilon_i)_K^{n+1-}. \tag{3.46}$$

The formulae (3.45) and (3.46) will be useful in the next section devoted to the time-implicit extension of the present procedure. To conclude the presentation of the explicit in time algorithm, let us observe from (3.41) that the right hand side of the generalized jump relations (3.45) is explicitly known so that the updated values $(X_i)_K^{n+1}$ are explicitly known as well.
The following statement will prove that the N entropies $(S_i)_K^{n+1}$ can be uniquely rebuilt from the knowledge of the $(N-1)$ values $(X_i)_K^{n+1}$ provided that the discrete solution has been kept in the physical space in the prediction step.

LEMMA 3.4. *Let be given* $(\rho, \rho\mathbf{w}, \rho E, (X_i)_{1\leq i\leq N-1}) \in \mathbb{R}^{d+N+1}$ *such that*

$$\rho > 0, \quad \rho\epsilon = \rho E - \frac{||\rho\mathbf{w}||^2}{2\rho} > 0. \tag{3.47}$$

Let $\psi : S_N \in \mathbb{R}_+^* \to \mathbb{R}$ *the nonlinear function defined by*

$$\begin{aligned} \psi(S_N) &= \big(\rho E - \frac{||\rho\mathbf{w}||^2}{2\rho}\big) - \sum_{i=1}^{N-1} \rho\epsilon_i\big(\rho, S_N \exp(X_i)\big) - \rho\epsilon_N(\rho, S_N) \\ &= \rho\epsilon - \sum_{i=1}^{N-1} \rho\epsilon_i\big(\rho, S_N \exp(X_i)\big) - \rho\epsilon_N(\rho, S_N). \end{aligned} \tag{3.48}$$

Then, ψ admits a unique zero $S_N^* \in \mathbb{R}_+^*$.
The proof of this lemma is given in [18].
Let us point out right now that the unique zero $S_N^* \in \mathbb{R}_+^*$ of (3.48) allows to define the $(N-1)$-uple $(S_i^*)_{1\leq i\leq N-1} \in (\mathbb{R}_+^*)^{N-1}$ setting

$$S_i^* = S_N^* \exp(X_i), \quad 1 \leq i \leq N-1. \tag{3.49}$$

So that by construction, the following consistency identity holds true :

$$\rho\epsilon = \rho E - \frac{||\rho\mathbf{w}||^2}{2\rho} = \sum_{i=1}^{N} \rho\epsilon_i(\rho, S_i^*). \tag{3.50}$$

Remark. It is worth noticing that the case of N polytropic pressure laws $p_i = (\gamma_i - 1)\rho\varepsilon_i$, $\gamma_i > 1$ being N given adiabatic coefficients, may lead to an explicit solution to $\psi(S_N) = 0$. Indeed, the particular choice

$$S_i = \frac{p_i}{\rho^{\gamma_i}}, \quad i = 1, ..., N, \tag{3.51}$$

gives

$$\psi(S_N) = 0 \Longleftrightarrow \rho\epsilon = \rho E - \frac{||\rho\mathbf{w}||^2}{2\rho} = S_N \sum_{i=1}^{N} \frac{\rho^{\gamma_i}}{\gamma_i - 1} \exp(X_i),$$

so that an explicit formula follows for S_N. Of course, the particular choice (3.51) does not imply the strict convexity property of the corresponding internal energies

$\varepsilon_i(\rho, S_i) = \rho^{\gamma_i} S_i/(\gamma_i - 1)$ but is however relevant in the sense that it leads to well-defined (*i.e.* positive) entropies $(S_i)_{1\leq i\leq N}$ provided that (3.47) holds true.

The last result of this section proves that the nonlinear correction procedure (3.39)-(3.40)-(3.45) is well-defined under the CFL condition (3.17) and defines an unique updated value $\mathbf{u}_K^{n+1}$ in the phase space $\Omega_{\mathbf{u}}$.

THEOREM 3.5. *Under the CFL condition (3.17), the total energy obeys in each cell K*

$$(\rho\epsilon)_K^{n+1} = (\rho E)_K^{n+1} - \frac{||(\rho \mathbf{w})_K^{n+1}||^2}{2\rho_K^{n+1}} > 0. \tag{3.52}$$

Therefore, there exists an unique N-uple $\{(S_i)_K^{n+1}\}_{1\leq i\leq N}$ with

$$(S_i)_K^{n+1} > 0 \quad \textit{for all } i = 1, ..., N, \tag{3.53}$$

and

$$\sum_{i=1}^{N} \rho_K^{n+1} \epsilon_i(\rho_K^{n+1}, (S_i)_K^{n+1}) = (\rho\epsilon)_K^{n+1}. \tag{3.54}$$

The proof of this theorem is given in [18].

3.3. Implicit in time numerical scheme. This section is motivated by the numerical approximation of the stationary solutions of the multi-pressure Euler equations in several space dimensions. Following the standard time-marching strategy (see [29] for instance), the stationary solutions are sought as the limit of unsteady solutions as time goes to infinity. For the sake of efficiency, one obviously needs to get rid of a CFL restriction to reach the steady state with the minimal CPU effort. Achieving this goal in practice requires a proper balance between the numerical cost of evaluation and the absence of CFL restriction. Such a balance is the main issue of this section devoted to the derivation of an efficient time-implicit formulation of the predictor-corrector procedure developed in the previous section. It will be seen hereafter that the correction step can receive minor and satisfactory modifications while the prediction step must be paid a particular attention.
According to a widely used procedure, the prediction step would be given a linearized time-implicit formulation of the finite volume scheme (3.8) :

$$\mathbf{u}_K^{n+1-} = \mathbf{u}_K^n - \frac{\Delta t}{|K|} \sum_{e\in\partial K} \mathcal{G}_{e,K}^{n,n+1-} |e|, \tag{3.55}$$

$$\mathcal{G}_{e,K}^{n,n+1-} = \mathcal{G}(\mathbf{u}_K^n, \mathbf{u}_{K_e}^n; \mathbf{n}_{e,K}) + \mathcal{L}_{e,K}^n(\mathbf{u}_K^{n+1-} - \mathbf{u}_K^n) + \mathcal{L}_{e,K_e}^n(\mathbf{u}_{K_e}^{n+1-} - \mathbf{u}_{K_e}^n) \tag{3.56}$$

for some matrices $\mathcal{L}_{e,K}^n$, $\mathcal{L}_{e,K_e}^n$ in $\mathcal{M}at(\mathbb{R}^{N+d+1})$. The flux formula (3.56) can be understood as some first order Taylor expansion in time of fully time-implicit numerical flux $\mathcal{G}(\mathbf{u}_K^{n+1-}, \mathbf{u}_{K_e}^{n+1-}; \mathbf{n}_{e,K})$. The mathematical justification of (3.55)-(3.56) seems to be out of reach but extensive applications in the classical Euler setting have grounded the efficiency of the procedure (see again [29]).
In the frame of the multi-pressure Euler equations, this standard strategy has however at least one important drawback. This drawback is related to the CPU effort associated with the resolution of the linear system (3.55) in the unknown $(\mathbf{u}_K^{n+1-})_{K\in\mathcal{T}_h}$

for large values of N. In presence of many specific entropies, this system can indeed become costly. In addition, it is important to keep in mind that by construction, the updates of the specific entropies $(S_i)_K^{n+1-}$, $i = 1, ..., N$, provided by the prediction step have to be recomputed in the correction step. Then, the more N is large, the more this important (extra) computational cost may appear superfluous.
To circumvent this drawback, we propose hereafter a far more efficient procedure based on a relaxation approach. At each time step, the procedure only requires to solve small size linear problems, namely N linear problems invoking the same scalar matrix for updating the specific entropies and a linear problem with elementary block matrices in $\mathcal{M}at(\mathbb{R}^{d+2})$ (i.e. independent of N) for the remaining unknowns. In addition, we will prove that the positivity of both the density ρ_K^{n+1-} and the total energy $\sum_{i=1}^N \rho_K^{n+1-}\epsilon_i(\rho_K^{n+1-}, (S_i)_K^{n+1-})$ is a sufficient (and necessary) condition for the correction procedure to take place. It turns convenient to first give a rough description of the proposed procedure to make easier its precise derivation.

3.3.1. A rough presentation of the time-implicit prediction step. We propose to derive at the prediction step an efficient linearized time-implicit finite volume method for approximating the solutions of

$$\left\{\begin{array}{l} \partial_t \rho + \nabla.\ (\rho \mathbf{w}) = 0, \\ \partial_t(\rho \mathbf{w}) + \nabla.\ (\rho \mathbf{w} \otimes \mathbf{w} + \displaystyle\sum_{i=1}^N p_i(\rho, S_i)\ \mathbf{I_d}) = 0, \\ \partial_t(\rho S_i) + \nabla.\ (\rho S_i \mathbf{w}) = 0, \quad i = 1, ..., N. \end{array}\right. \tag{3.57}$$

The efficiency of the method will be achieved when suitably modifying the nonlinearities in the PDEs (3.57) by means of a relaxation approximation. The relaxation strategy under consideration in this paper is motivated by the works of Suliciu [39], Bouchut [9], Chalons and Coquel [13], Coquel *et al.* [22] and Chalons *et al.* [17]. The idea is to modify only the total pressure law which concentrates the main nonlinearities in the original equations. More precisely, the pressure $\sum_{i=1}^N p_i(\rho, S_i)$ is no longer understood as a nonlinear function of the unknowns ρ, $\rho\mathbf{w}$ and $(S_i)_{1\leq i\leq N}$ but is now handled with a new unknown Π evolving according to its own partial differential equation. This new unknown is subject to a relaxation procedure which purpose is to restore the original pressure law in the regime of an infinite relaxation rate $\lambda > 0$. The relaxation PDE model reads :

$$\left\{\begin{array}{l} \partial_t \rho^\lambda + \nabla.\ (\rho \mathbf{w})^\lambda = 0, \\ \partial_t(\rho \mathbf{w})^\lambda + \nabla.\ (\rho \mathbf{w} \otimes \mathbf{w} + \Pi\ \mathbf{I_d})^\lambda = 0, \\ \partial_t(\rho \Pi)^\lambda + \nabla.\ \big((\rho\Pi + a^2)\mathbf{w}\big)^\lambda = \lambda \rho^\lambda (\displaystyle\sum_{i=1}^N p_i(\rho^\lambda, S_i^\lambda) - \Pi^\lambda), \\ \partial_t(\rho S_i)^\lambda + \nabla.\ (\rho S_i \mathbf{w})^\lambda = 0, \quad i = 1, ..., N. \end{array}\right. \tag{3.58}$$

Here a is a given positive real number to be precised later on. Clearly when λ goes to $+\infty$, Π^λ formally coincides with the original pressure law :

$$\lim_{\lambda \to +\infty} \Pi^\lambda = \sum_{i=1}^N p_i(\rho, S_i). \tag{3.59}$$

In order to prevent some instabilities in the asymptotic regime $\lambda \to +\infty$, it is well-known that the so-called subcharacteristic conditions (or Whitham condition) are

expected to be satisfied. We refer to the work by Liu [36] and Chen, Levermore and Liu [24]. These subcharacteristic conditions are satisfied provided that the free coefficient $a > 0$ in (3.58) upper-bounds the exact Lagrangian sound speed ρc, namely :

$$a > \rho \times c(\rho, (S_i)_{1\leq i\leq N}) \quad \text{with} \quad c^2(\rho, (S_i)_{1\leq i\leq N}) = \sum_{i=1}^{N} \partial_\rho p_i(\rho, S_i), \tag{3.60}$$

for all the states under consideration. Again, the reader is referred to the works [9], [13], [22], [17] for a detailed discussion of (3.60) and its relationship with the validity of entropy inequalities.

Let us now highlight the interest of the relaxation PDE model (3.58) in the derivation of an efficient time-implicit method. To that purpose, it is convenient to set (right temporarily) the relaxation parameter λ to zero to observe that the $d+2$ governing equations associated with the density ρ, the momentum $\rho\mathbf{w}$ and the relaxation pressure $\rho\Pi$ are decoupled from the N remaining ones. In other words, the nonlinear coupling of the $d+2$ equations with the N specific entropies $(S_i)_{1\leq i\leq N}$ solely takes place via the relaxation source term $\lambda\rho^\lambda(\sum_{i=1}^N p_i(\rho^\lambda, S_i^\lambda) - \Pi^\lambda)$.

We propose to take advantage of this weak coupling by splitting the evolution in time of the specific entropies $(S_i)_{1\leq i\leq N}$ and of $(\rho, \rho\mathbf{w}, \rho\Pi)$. More precisely, let be given $\mathbf{u}_h$ a discrete approximate solution of the *equilibrium* system (3.57). To update this approximate solution, we propose to solve the relaxation model (3.58) in the asymptotic regime $\lambda \to +\infty$ with an initial data built at equilibrium from $\mathbf{u}_h(\mathbf{x}, t^n)$ by setting

$$(\rho\Pi)_h(\mathbf{x}, t^n) = \rho_h(\mathbf{x}, t^n) \sum_{i=1}^{N} p_i\big(\rho_h(\mathbf{x}, t^n), (S_i)_h(\mathbf{x}, t^n)\big).$$

The proposed splitting strategy operates as follows:

First step
Solve in the slab $[0, \Delta t]$ the N advection equations

$$\left\{ \begin{array}{l} \partial_t S_i + \mathbf{w}.\nabla S_i = 0, \quad i = 1, ..., N, \\ S_i(\mathbf{x}, 0) = (S_i)_h(\mathbf{x}, t^n), \end{array} \right. \tag{3.61}$$

and denote $(S_i)_h(\mathbf{x}, t^{n+1-})$ the resulting prediction at time $t^n + \Delta t$.

Second step
Solve in the slab $[0, \Delta t]$ the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t \rho^\lambda + \nabla.\ (\rho\mathbf{w})^\lambda = 0, \\ \partial_t (\rho\mathbf{w})^\lambda + \nabla.\ (\rho\mathbf{w} \otimes \mathbf{w} + \Pi\ \mathbf{I_d})^\lambda = 0, \\ \partial_t (\rho\Pi)^\lambda + \nabla.\ ((\rho\Pi + a^2)\mathbf{w})^\lambda = \lambda\Big(\rho^\lambda \sum_{i=1}^{N} p_i\big(\rho^\lambda, (S_i)_h(\mathbf{x}, t^{n+1-})\big) - (\rho\Pi)^\lambda\Big) \end{array} \right. \tag{3.62}$$

with initial data $\big(\rho_h(\mathbf{x}, t^n), (\rho\mathbf{w})_h(\mathbf{x}, t^n), (\rho\Pi)_h(\mathbf{x}, t^n)\big)$ in the regime $\lambda \to +\infty$. In agreement with (3.60), the constant a is chosen so that

$$a > \rho \times c(\rho, ((S_i)_h^{n+1-})_{1\leq i\leq N}). \tag{3.63}$$

3.3.2. The advection problem (3.61) in details. In order to predict the evolution of the N specific entropies $(S_j)_{j=1,...,N}$, we propose the following time-implicit update formula directly motivated by the time-explicit analogous (3.27)-(3.29) :

$$\begin{aligned}(S_j)_K^{n+1-} &= (S_j)_K^n - \tfrac{\Delta t}{|K|}\sum_{e\in\partial K}|e|\,\big\{(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,+}^n\big((S_j)_{e,K}^{n+1-}-(S_j)_K^{n+1-}\big)+\\ &\qquad (\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,-}^n\big((S_j)_{e,K}^{n+1-}-(S_j)_K^{n+1-}\big)\big\}\end{aligned} \tag{3.64}$$

$$\text{with}\quad (S_j)_{e,K}^{n+1-} = \begin{cases}(S_j)_K^{n+1-} & \text{if} \quad (\mathbf{w}\cdot\mathbf{n}_{e,K})_e^n\geq 0,\\ (S_j)_{K_e}^{n+1-} & \text{otherwise.}\end{cases} \tag{3.65}$$

Here, $(\mathbf{w}\cdot\mathbf{n}_{e,K})_e^n$ denotes a given consistent approximation of the velocity $\mathbf{w}\cdot\mathbf{n}_{e,K}$ at each time t^n and on each face e of each polyhedron K, say for instance

$$(\mathbf{w}\cdot\mathbf{n}_{e,K})_e^n = \frac{1}{2}(\mathbf{w}_K^n+\mathbf{w}_{K_e}^n)\cdot\mathbf{n}_{e,K}, \tag{3.66}$$

and $(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,\pm}^n$ the corresponding nonpositive and nonnegative parts. An equivalent formulation of (3.64)-(3.65) is

$$\begin{aligned}(S_j)_K^n &= \Big(1-\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,-}^n\,|e|\,\Big)(S_j)_K^{n+1-}\\ &+\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,-}^n\,|e|\,(S_j)_{K_e}^{n+1-},\quad j=1,...,N,\end{aligned} \tag{3.67}$$

which shows that the N linear systems to be solved to find the set of entropies $\big((S_j)_K^{n+1-}\big)_{K\in\mathcal{T}_h}$ involve the same linear operator (say the same matrix) for each $j=1,...,N$. Of course, this property is very pleasant and give rise to well-designed procedures in order to solve (3.64) (or (3.67)) efficiently. We refer for instance the reader to [32] for more details on this point.
In addition, the following stability result holds true :

LEMMA 3.6. *For each time step Δt, the time-implicit update formula (3.64) (or equivalently (3.67)) of the unknowns $\big((S_j)_K^{n+1-}\big)_{K\in\mathcal{T}_h}$ admits for each $j=1,...,N$ an unique solution satisfying the following maximum principle :*

$$\min_{K\in\mathcal{T}_h}(S_j)_K^n \leq (S_j)_K^{n+1-}\leq \max_{K\in\mathcal{T}_h}(S_j)_K^n,\quad j=1,...,N,\quad K\in\mathcal{T}_h.$$

Proof. The common matrix involved in (3.67) clearly is strictly diagonal dominent whereas the diagonal (respectively extra-diagonal) terms are all positive (resp. nonpositive). Then, this matrix is invertible which ensures existence and uniqueness on the one hand, and importantly, the inverse matrix has only nonnegative terms on the other hand (see for instance [32]). In order to derive the maximum principle property, let us observe that for any given constant C the following equality holds

$$\begin{aligned}(S_j)_K^n - C &= \big(1-\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,-}^n\,|e|\,\big)\big\{(S_j)_K^{n+1-}-C\big\}\\ &+\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\mathbf{w}\cdot\mathbf{n}_{e,K})_{e,-}^n\,|e|\,\big\{(S_j)_{K_e}^{n+1-}-C\big\},\quad j=1,...,N.\end{aligned} \tag{3.68}$$

Then, it suffices to choose respectively $C=\max_{K\in\mathcal{T}_h}(S_j)_K^n$ and $C=\min_{K\in\mathcal{T}_h}(S_j)_K^n$, arguing about the nonnegativity of the inverse matrix coefficients. □

3.3.3. The hydrodynamic subsystem (3.62) in details. System (3.62) is first given the condensed notation

$$\partial_t \mathbf{v}^\lambda + \nabla.\ \mathbf{F}(\mathbf{v}^\lambda) = \lambda \mathcal{R}(\mathbf{v}^\lambda, \big((S_j)_h(\mathbf{x}, t^{n+1-})\big)_{1\leq j\leq N}), \tag{3.69}$$

with $\mathbf{v} = (\rho, \rho\mathbf{w}, \rho\Pi)$, $\mathcal{R}(\mathbf{v}) = \Big(0, 0_{\mathbb{R}^d}, \rho\sum_{i=1}^N p_i(\rho, (S_i)_h(\mathbf{x}, t^{n+1-})) - \rho\Pi\Big)$, and clear definitions for the vector-valued function $\mathbf{F}(\mathbf{v}) = (\mathcal{F}_{x_j}(\mathbf{v}))_{1\leq j\leq d}$.
Let us begin by briefly reporting some useful properties of (3.69). First, these equations have the rotational invariance property. Then, for any given unit vector $\mathbf{n} = {}^t(n_1, ..., n_d)$ of $\mathbb{R}^d$, the matrix $\nabla_{\mathbf{v}}\big(\sum_{j=1}^d \mathcal{F}_{x_j}(\mathbf{v})\, n_j\big)$ is $\mathbb{R}-$diagonalizable over the natural phase space

$$\Omega_{\mathbf{v}} = \big\{(\rho, \rho\mathbf{w}, \rho\Pi) \in \mathbb{R}^{d+2} / \rho > 0, \rho\mathbf{w} \in \mathbb{R}^d, \rho\Pi \in \mathbb{R}\big\}, \tag{3.70}$$

with eigenvalues

$$\lambda_1(\mathbf{v}, \mathbf{n}) = \mathbf{w}.\mathbf{n} - \frac{a}{\rho} < \lambda_{j=2,...,d+1}(\mathbf{v}, \mathbf{n}) = \mathbf{w}.\mathbf{n} < \lambda_{d+2}(\mathbf{v}, \mathbf{n}) = \mathbf{w}.\mathbf{n} + \frac{a}{\rho}.$$

The first order system extracted from (3.69) is then hyperbolic. Importantly, all the associated characteristic fields are linearly degenerate, meaning poorly speaking that all the propagating waves behave as linear waves.
Let us now turn to the numerical approximation of the solutions of (3.69) in the asymptotic regime $\lambda \to \infty$. Before addressing the time-implicit issue, it is worthy to briefly review the time-explicit approach.
A *time-explicit* numerical scheme is usually obtained by applying a splitting strategy to (3.69). It consists in solving (3.69) with $\lambda = 0$ in a first step, and the ODE problem

$$\partial_t \mathbf{v}^\lambda = \lambda \mathcal{R}(\mathbf{v}^\lambda, \big((S_j)_h(\mathbf{x}, t^{n+1-}\big)_{1\leq j\leq N})$$

in the limit $\lambda \to \infty$ in a second step. According to (3.8) and arguing about the rotational invariance of the equations, the first step amounts to set

$$\mathbf{v}_K^{n+1-} = \mathbf{v}_K^n - \frac{\Delta t}{|K|}\sum_{e\in\partial K} \mathcal{F}_{e,K}^n |e|$$

where the definition of $\mathcal{F}_{e,K}^n$ follows from the following quasi-1D homogeneous form of (3.69) :

$$\partial_t \mathbf{v} + \partial_x \mathbf{F}_{i_1}(\mathbf{v}) = 0, \tag{3.71}$$

with

$$\mathbf{F}_{i_1}(\mathbf{v}) = \Big(\rho w_1, \rho w_1^2 + \Pi, (\rho w_1 w_j)_{2\leq j\leq d}, (\rho\Pi + a^2) w_1\Big). \tag{3.72}$$

Here again i_1 denotes the first unit vector of the canonical basis of $\mathbb{R}^d$ and $\mathbf{F}_{i_1} = \mathcal{F}_{x_1}$. More precisely, using the same notations as in section 3.2 we have

$$\mathcal{F}_{e,K}^n = \mathcal{F}(\mathbf{v}_K^n, \mathbf{v}_{K_e}^n; \mathbf{n}_{e,K}) = T_{e,K}^{-1}\ \mathcal{F}(T_{e,K}\mathbf{v}_K^n, T_{e,K}\mathbf{v}_{K_e}^n; i_1). \tag{3.73}$$

The linear degeneracy property of (3.71) makes easy the calculation of the Riemann solution $\mathcal{W}(\frac{x}{t}; \mathbf{v}_L, \mathbf{v}_R)$ associated with the initial data

$$\mathbf{v}_0(x) = \mathbf{v}_L \text{ if } x < 0,\ \mathbf{v}_R \text{ if } x > 0, \tag{3.74}$$

with $\mathbf{v}_L$ and $\mathbf{v}_R$ in $\Omega_{\mathbf{v}}$ (see Proposition 3.7). This suggests to set

$$\mathcal{F}(\mathbf{v}_L, \mathbf{v}_R; i_1) = \mathbf{F}_{i_1}\big(\mathcal{W}(0^+; \mathbf{v}_L, \mathbf{v}_R)\big). \tag{3.75}$$

PROPOSITION 3.7. *Let be given two N-uples of specific entropies $(S_{iL})_{1\leq i\leq N}$ and $(S_{iR})_{1\leq i\leq N}$ and two states $\mathbf{v}_L$ and $\mathbf{v}_R$ in $\Omega_{\mathbf{v}}$ that we assume to be at equilibrium, that is such that $\Pi_L = \mathcal{P}_L$ and $\Pi_R = \mathcal{P}_R$. Let us define the constant a in (3.72) according to (3.63), namely*

$$a(\mathbf{v}_L, \mathbf{v}_R) > \max\big(\rho_L c(\rho_L, (S_{iL})_{1\leq i\leq N}), \rho_R c(\rho_R, (S_{iR})_{1\leq i\leq N})\big), \tag{3.76}$$

with

$$\begin{aligned} \sigma_1(\mathbf{v}_L, \mathbf{v}_R) = w_{1,L} - \frac{a(\mathbf{v}_L, \mathbf{v}_R)}{\rho_L} < \sigma_2(\mathbf{v}_L, \mathbf{v}_R) = w_1^\star(\mathbf{v}_L, \mathbf{v}_R) \\ \sigma_2(\mathbf{v}_L, \mathbf{v}_R) < \sigma_3(\mathbf{v}_L, \mathbf{v}_R) = w_{1,R} + \frac{a(\mathbf{v}_L, \mathbf{v}_R)}{\rho_R}, \end{aligned} \tag{3.77}$$

and

$$w_1^\star(\mathbf{v}_L, \mathbf{v}_R) = \frac{1}{2}(w_{1,R} + w_{1,L}) - \frac{1}{2a(\mathbf{v}_L, \mathbf{v}_R)}(\mathcal{P}_R - \mathcal{P}_L). \tag{3.78}$$

The self-similar solution $\mathcal{W}(.; \mathbf{v}_L, \mathbf{v}_R)$ of the Cauchy problem (3.71) with initial data (3.74) is made of four constant states $\mathbf{v}_L, \mathbf{v}_1(\mathbf{v}_L, \mathbf{v}_R), \mathbf{v}_2(\mathbf{v}_L, \mathbf{v}_R), \mathbf{v}_R$ separated by three contact discontinuities propagating with speeds $\sigma_i(\mathbf{v}_L, \mathbf{v}_R)$, $i = 1, 2, 3$ respectively :

$$\mathcal{W}(x/t; \mathbf{v}_L, \mathbf{v}_R) = \begin{cases} \mathbf{v}_L & \text{if} \quad \frac{x}{t} < \sigma_1(\mathbf{v}_L, \mathbf{v}_R), \\ \mathbf{v}_1(\mathbf{v}_L, \mathbf{v}_R) & \text{if} \quad \sigma_1(\mathbf{v}_L, \mathbf{v}_R) < \frac{x}{t} < \sigma_2(\mathbf{v}_L, \mathbf{v}_R), \\ \mathbf{v}_2(\mathbf{v}_L, \mathbf{v}_R) & \text{if} \quad \sigma_2(\mathbf{v}_L, \mathbf{v}_R) < \frac{x}{t} < \sigma_3(\mathbf{v}_L, \mathbf{v}_R), \\ \mathbf{v}_R & \text{if} \quad \sigma_3(\mathbf{v}_L, \mathbf{v}_R) < \frac{x}{t}. \end{cases} \tag{3.79}$$

The intermediate states $\mathbf{v}_1(\mathbf{v}_L, \mathbf{v}_R)$ and $\mathbf{v}_2(\mathbf{v}_L, \mathbf{v}_R)$ belong to the phase space $\Omega_{\mathbf{v}}$ (i.e. $\rho_1(\mathbf{v}_L, \mathbf{v}_R) > 0$, $\rho_2(\mathbf{v}_L, \mathbf{v}_R) > 0$) and are recovered from the next formulas with $a = a(\mathbf{v}_L, \mathbf{v}_R)$:

$$\begin{aligned} &\Pi^\star = \Pi_1 = \Pi_2 = \frac{1}{2}(\mathcal{P}_L + \mathcal{P}_R) - \frac{a}{2}(w_{1,R} - w_{1,L}), \\ &w_{1,1} = w_{1,2} = w_1^\star, \\ &w_{j,1} = w_{j,L}, \quad w_{j,2} = w_{j,R}, \quad 2 \leq j \leq d \\ &\frac{1}{\rho_1} = \frac{1}{\rho_L} - \frac{1}{a}(w_{1,L} - w_1^\star), \\ &\frac{1}{\rho_2} = \frac{1}{\rho_R} - \frac{1}{a}(w_1^\star - w_{1,R}), \end{aligned} \tag{3.80}$$

where $w_1^\star = w_1^\star(\mathbf{v}_L, \mathbf{v}_R)$ is given in (3.78). The derivation of a *time-implicit* numerical scheme is much more challenging. The proposed discretization here strongly relies on the previous work [17] devoted to the time-implicit formulations of relaxation schemes for the usual Euler equations. It is first shown in [17] that splitting the relaxation source term $\lambda\mathcal{R}(\mathbf{v}^\lambda, ((S_j)_h(\mathbf{x}, t^{n+1-})_{1\leq j\leq N})$ and the flux divergence $\nabla.\,\mathbf{F}(\mathbf{v}^\lambda)$ cannot result in a well-balanced approximation of the stationary solutions of (3.69), namely satisfying $\nabla.\,\mathbf{F}(\mathbf{v}^\lambda) = \lambda\mathcal{R}(\mathbf{v}^\lambda, \big((S_j)_h(\mathbf{x}, t^{n+1-})\big)_{1\leq j\leq N})$. On the

contrary, a relevant time-implicit procedure requires to handle simultaneously the relaxation source term and the flux divergence. We are thus led to consider an update formula of the form

$$\mathbf{v}_K^{n+1-} = \mathbf{v}_K^n - \frac{\Delta t}{|K|}\sum_{e\in\partial K}\mathcal{F}_{e,K}^{n,n+1-}|e| + \lambda\Delta t\mathcal{R}_K^{n+1-} \tag{3.81}$$

in the limit $\lambda \to \infty$. The rest of this section aims at defining the numerical fluxes $\mathcal{F}_{e,K}^{n,n+1-}$ and the discrete relaxation source term $\mathcal{R}_K^{n+1-}$.
For the sake of efficiency, the definition of $\mathcal{F}_{e,K}^{n+1-}$ is classically based on a linearization operation. The key point in this process is the existence of a Roe-matrix for equivalently re-expressing the Godunov numerical flux (3.75). More precisely, we have the following result whose proof is postponed in the next subsection. Note from now on that this proof will be of crucial interest in the forthcoming correction step.

THEOREM 3.8. *For any given pair of states $(\mathbf{v}_L, \mathbf{v}_R) \in \Omega_{\mathbf{v}}^2$, there exists a Roe-matrix $\mathbf{A}_{i_1}(\mathbf{v}_L, \mathbf{v}_R)$ for the quasi-1D relaxation system (3.71), namely satisfying*

$$\begin{array}{ll} \text{(i)} & \mathbf{A}_{i_1}(\mathbf{v},\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{F}_{i_1}(\mathbf{v}), \\ \text{(ii)} & \mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)\ (\mathbf{v}_R-\mathbf{v}_L) = \mathbf{F}_{i_1}(\mathbf{v}_R) - \mathbf{F}_{i_1}(\mathbf{v}_L), \\ \text{(iii)} & \mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R) \quad \textit{is } \mathbb{R}\textit{-diagonalizable}, \end{array} \tag{3.82}$$

and such that the Godunov numerical flux function (3.75) is algebraically equivalent to the following Roe numerical flux function :

$$\mathbf{F}_{i_1}(\mathcal{W}(0^+;\mathbf{v}_L,\mathbf{v}_R)) = \frac{1}{2}\Big(\mathbf{F}_{i_1}(\mathbf{v}_L) + \mathbf{F}_{i_1}(\mathbf{v}_R) - |\mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)|(\mathbf{v}_R-\mathbf{v}_L)\Big). \tag{3.83}$$

According to (3.73), (3.75) and (3.83) we then classically set

$$\begin{aligned} \mathcal{F}_{e,K}^{n,n+1-} &= T_{e,K}^{-1}\mathbf{F}_{i_1}(\mathcal{W}(0^+;T_{e,K}\mathbf{v}_K^n, T_{e,K}\mathbf{v}_{K_e}^n)) \\ &+\frac{1}{2}T_{e,K}^{-1}(\nabla_{\mathbf{v}}\ \mathbf{F}_{i_1}(T_{e,K}\mathbf{v}_K^n) + |\mathbf{A}_{i_1}(T_{e,K}\mathbf{v}_K^n, T_{e,K}\mathbf{v}_{K_e}^n)|)T_{e,K}\ \delta(\mathbf{v}_K^n) \\ &+\frac{1}{2}T_{e,K}^{-1}(\nabla_{\mathbf{v}}\ \mathbf{F}_{i_1}(T_{e,K}\mathbf{v}_{K_e}^n) - |\mathbf{A}_{i_1}(T_{e,K}\mathbf{v}_K^n, T_{e,K}\mathbf{v}_{K_e}^n)|)T_{e,K}\ \delta(\mathbf{v}_{K_e}^n), \end{aligned} \tag{3.84}$$

where the time increments are defined by $\delta(\mathbf{v}^n) = \mathbf{v}^{n+1-} - \mathbf{v}^n$. Regarding now the discrete relaxation source term $\mathcal{R}_K^{n+1-}$ we propose here again to linearize the fully implicit expected expression

$$\mathcal{R}(\mathbf{v}_K^{n+1-}, \big((S_j)_K^{n+1-}\big)_{1\le j\le N}) \tag{3.85}$$

in order to soften the computational complexity. More precisely, we set

$$\mathcal{R}_K^{n+1-} = \mathcal{R}(\mathbf{v}_K^n, \big((S_j)_K^{n+1-}\big)_{1\le j\le N}) + \big(0, \mathbf{0}, \sum_{i=1}^{N}\frac{\partial}{\partial\rho}(\rho p_i)\big(\rho_K^n, (S_i)_K^{n+1-}\big)\times\delta(\rho_K^n) - \delta(\rho\Pi)_K^n\big)^t$$

where $\delta(\rho_K^n) = \rho_K^{n+1-} - \rho_K^n$ and $\delta(\rho\Pi)_K^n = (\rho\Pi)_K^{n+1-} - (\rho\Pi)_K^n$. This is nothing but a first-order Taylor expansion of the last component of (3.85), based on its definition

provided by (3.62).
With such a definition, letting λ go to $+\infty$ in the last component of (3.81) yields

$$\sum_{i=1}^{N} \frac{\partial}{\partial \rho}(\rho p_i)\big(\rho_K^n, (S_i)_K^{n+1-}\big) \times \delta(\rho_K^n) - \delta(\rho\Pi)_K^n = -\Big(\sum_{i=1}^{N} \rho_K^n p_i\big(\rho_K^n, (S_i)_K^{n+1-}\big) - (\rho\Pi)_K^n\Big), \tag{3.86}$$

that is a linear equation to define the increment $\delta(\rho\Pi)_K^n$ with respect to $\delta(\rho_K^n)$.
To sum up, solving (3.81) in the limit $\lambda \to +\infty$ classically amounts to find $(\mathbf{v}_K^{n+1-})_{K\in\mathcal{T}_h}$ as the solution of a linear problem in the time increment $\delta(\mathbf{v}_K^n) = (\mathbf{v}_K^{n+1-} - \mathbf{v}_K^n)$ of the following form

$$(\mathbf{v}_K)^{n+1-} = (\mathbf{v}_K)^{\,n} - \frac{\Delta t}{|K|} \sum_{e\in\partial K} \overline{\mathcal{F}}_{e,K}^{n,\,n+1-} |e| - \mathcal{S}_K^{n,\,n+1-}, \tag{3.87}$$

with $\overline{\mathcal{F}}_{e,K}^{n,\,n+1-} = \mathcal{F}(\mathbf{v}_K^n, \mathbf{v}_{K_e}^n; \mathbf{n}_{e,K}) + \overline{\mathcal{L}}_{e,K}^n \delta(\mathbf{v}_K^n) + \overline{\mathcal{L}}_{e,K_e}^n \delta(\mathbf{v}_{K_e}^n)$, where the matrices $\overline{\mathcal{L}}_{e,K}^n,\ \overline{\mathcal{L}}_{e,K_e}^n \in \mathcal{M}at(\mathbb{R}^{d+2})$ and the source term $\mathcal{S}_K^{n,\,n+1-}$ are respectively inferred from the formulas (3.81), (3.84) and when modifying the last component of (3.81) so as to obey the "equilibrium" relation (3.86). The details are left to the reader (see also [17]).

3.3.4. Proof of Theorem 3.8. The proof is detailed in [18]. We just mention here that it does use a similar result already established in [17] and recalled here. This result is concerned with the following system

$$\left\{\begin{array}{ll} \partial_t \rho + \partial_x(\rho w_1) = 0, & \\ \partial_t(\rho w_1) + \partial_x(\rho w_1^2 + \Pi) = 0, & \\ \partial_t(\rho w_j) + \partial_x(\rho w_1 w_j) = 0, & j = 2, ..., d, \\ \partial_t(\rho\Pi) + \partial_x(\rho\Pi + a^2) w_1 = 0, & \\ \partial_t(\rho E) + \partial_x(\rho E + \Pi) w_1 = 0, & \end{array}\right. \tag{3.88}$$

which turns out to be very close to (3.71). Actually, the only difference comes from the addition of a new equation that mimics the classical evolution of the total energy ρE. Observe however that the latter is utterly decoupled from the first $(d+2)$ equations (ρE does not appear in these equations). We will use the following condensed notations for (3.88) :

$$\partial_t \overline{\mathbf{v}} + \partial_x \overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}) = 0 \tag{3.89}$$

with $\overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}) = \Big(\rho w_1, \rho w_1^2 + \Pi, (\rho w_1 w_j)_{2\le j\le d}, (\rho\Pi + a^2) w_1, (\rho E + \Pi) w_1\Big)$.
In [17], the Riemann solution $\overline{\mathcal{W}}(\frac{x}{t}; \overline{\mathbf{v}}_L, \overline{\mathbf{v}}_R)$ associated with (3.89) and the initial data

$$\overline{\mathbf{v}}_0(x) = \overline{\mathbf{v}}_L \text{ if } x < 0,\ \overline{\mathbf{v}}_R \text{ if } x > 0, \tag{3.90}$$

for two constant states $\overline{\mathbf{v}}_L$ and $\overline{\mathbf{v}}_R$ is given, and the following equivalence between a Godunov and Roe numerical flux function is proved.

THEOREM 3.9. *There exists a Roe-matrix $\overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L, \overline{\mathbf{v}}_R)$ for the quasi-1D relaxation system (3.89), namely satisfying*

$$\begin{array}{ll} \text{(i)} & \overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}, \overline{\mathbf{v}}) = \nabla_{\overline{\mathbf{v}}} \overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}), \\ \text{(ii)} & \overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L, \overline{\mathbf{v}}_R)\ (\overline{\mathbf{v}}_R - \overline{\mathbf{v}}_L) = \overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}_R) - \overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}_L), \\ \text{(iii)} & \overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L, \overline{\mathbf{v}}_R) \quad \textit{is } \mathbb{R}\textit{-diagonalizable}, \end{array} \tag{3.91}$$

and such that the Godunov numerical flux function $\overline{\mathbf{F}}_{i_1}\big(\overline{\mathcal{W}}(0^+;\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)\big)$ *is algebraically equivalent to the following Roe numerical flux function :*

$$\overline{\mathbf{F}}_{i_1}(\overline{\mathcal{W}}(0^+;\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)) = \frac{1}{2}\Big(\overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}_L)+\overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}}_R)-|\overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)|(\overline{\mathbf{v}}_R-\overline{\mathbf{v}}_L)\Big). \quad (3.92)$$

The existence of the Roe-matrix $\mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)$ such that (3.82) and (3.83) hold true follows from the existence of $\overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)$ (see [18] for the details).

Remark. The first $(d+2)$ components of $\overline{\mathbf{F}}_{i_1}(\overline{\mathbf{v}})$ and of $\overline{\mathcal{W}}(0^+;\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)$ do not depend on the unknown ρE (which highlights the decoupling of the total energy) so that it is actually expected from (3.92) that $|\overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)|$, and thus $\overline{\mathbf{A}}_{i_1}(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)$, has the form

$$\begin{pmatrix} |\mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)| & \mathbf{0} \\ \mathbf{Y} & Z \end{pmatrix} \quad (3.93)$$

with $\mathbf{Y}=\mathbf{Y}(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)$, $Z=Z(\overline{\mathbf{v}}_L,\overline{\mathbf{v}}_R)$ and $\mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)$ is the Roe matrix involved in (3.83) :

$$\mathbf{F}_{i_1}(\mathcal{W}(0^+;\mathbf{v}_L,\mathbf{v}_R)) = \frac{1}{2}\Big(\mathbf{F}_{i_1}(\mathbf{v}_L)+\mathbf{F}_{i_1}(\mathbf{v}_R)-|\mathbf{A}_{i_1}(\mathbf{v}_L,\mathbf{v}_R)|(\mathbf{v}_R-\mathbf{v}_L)\Big).$$

3.3.5. The implicit in time correction step in details. As already stated, the implicit in time correction step can be defined from its explicit in time version with only a few (natural) modifications.
First, according to (3.39), we propose to keep the density and momentum values obtained in the prediction step (see subsection 3.3.3 above) :

$$\rho_K^{n+1}=\rho_K^{n+1-}, \quad (\rho\mathbf{w})_K^{n+1}= \quad (\rho\mathbf{w})_K^{n+1-}. \quad (3.94)$$

Regarding then the total energy ρE, an implicit version of (3.40) has to be proposed, namely a conservative update formula of the form

$$(\rho E)_K^{n+1}=(\rho E)_K^n-\frac{\Delta t}{|K|}\sum_{e\in\partial K}(\rho\mathcal{H}\mathbf{w})_{e,K}^{n+1}|e| \quad (3.95)$$

with $(\rho\mathcal{H}\mathbf{w})_{e,K}^{n+1}$ to be precised. In that aim, let us first observe that thanks to (3.92)-(3.93), a natural definition of the *explicit* numerical flux $(\rho\mathcal{H}\mathbf{w})_{e,K}^n$ based on the relaxation system (3.88) (or (3.89) equivalently) is given by (see also (3.19))

$$\begin{aligned}
(\rho\mathcal{H}\mathbf{w})_{e,K}^n &= \big((\rho E+\Pi)w_1\big)\big(\overline{\mathcal{W}}(0^+;T_{e,K}\overline{\mathbf{v}}_K^n,T_{e,K}\overline{\mathbf{v}}_{K_e}^n)\big)\\
&= \frac{1}{2}\big((\rho E+\Pi)w_1\big)\big(T_{e,K}\overline{\mathbf{v}}_K^n\big)+\frac{1}{2}\big((\rho E+\Pi)w_1\big)\big(T_{e,K}\overline{\mathbf{v}}_{K_e}^n\big)\\
&- \frac{1}{2}\mathbf{Y}(T_{e,K}\overline{\mathbf{v}}_K^n,T_{e,K}\overline{\mathbf{v}}_{K_e}^n)\,.\,T_{e,K}\big(\mathbf{v}_{K_e}^n-\mathbf{v}_K^n\big)\\
&- \frac{1}{2}Z(T_{e,K}\overline{\mathbf{v}}_K^n,T_{e,K}\overline{\mathbf{v}}_{K_e}^n)\times\big((\rho E)_{K_e}^n-(\rho E)_K^n\big).
\end{aligned}$$

In perfect agreement with (3.84), we then propose to make implicit this formula by performing a first order Taylor expansion of the first two terms and linearizing the other two by freezing the coefficients $\mathbf{Z}$ and Y at time t^n :

$$\begin{aligned}(\rho\mathcal{H}\mathbf{w})_{e,K}^{n+1} &= (\rho\mathcal{H}\mathbf{w})_{e,K}^{n} \\ &+ \frac{1}{2}\nabla_{\overline{\mathbf{v}}}\big((\rho E+\Pi)w_1\big)T_{e,K}\delta\big(\overline{\mathbf{v}}_K^n\big) + \frac{1}{2}\nabla_{\overline{\mathbf{v}}}\big((\rho E+\Pi)w_1\big)T_{e,K}\delta\big(\overline{\mathbf{v}}_{K_e}^n\big) \\ &- \frac{1}{2}\mathbf{Y}(T_{e,K}\overline{\mathbf{v}}_K^n, T_{e,K}\overline{\mathbf{v}}_{K_e}^n)\,.\,T_{e,K}\big(\delta(\mathbf{v}_{K_e}^n)-\delta(\mathbf{v}_K^n)\big) \\ &- \frac{1}{2}Z(T_{e,K}\overline{\mathbf{v}}_K^n, T_{e,K}\overline{\mathbf{v}}_{K_e}^n)\times\big(\delta(\rho E)_{K_e}^n-\delta(\rho E)_K^n\big),\end{aligned} \tag{3.96}$$

where the time increments are defined by

$$\delta(\overline{\mathbf{v}}_K^n) = \begin{pmatrix} \delta(\mathbf{v}_K^n) \\ \delta(\rho E)_K^n \end{pmatrix} = \begin{pmatrix} \delta(\mathbf{v}_K^n) \\ (\rho E)_K^{n+1}-(\rho E)_K^n \end{pmatrix},$$

with

$$\delta(\mathbf{v}_K^n) = \begin{pmatrix} \rho_K^{n+1-}-\rho_K^n \\ (\rho\mathbf{w})_K^{n+1-}-(\rho\mathbf{w})_K^n \\ (\rho\Pi)_K^{n+1-}-(\rho\Pi)_K^n \end{pmatrix} = \begin{pmatrix} \rho_K^{n+1}-\rho_K^n \\ (\rho\mathbf{w})_K^{n+1}-(\rho\mathbf{w})_K^n \\ (\rho\Pi)_K^{n+1-}-(\rho\Pi)_K^n \end{pmatrix}.$$

Recall that the time increments $\delta(\mathbf{v}_K^n)$ are known by the prediction step so that $\delta(\rho E)_K^n$ is actually the only unknown in the scalar and linear system (3.95)-(3.96). Once $(\rho E)_K^{n+1}$ is computed, the total internal energy $(\rho\epsilon)_K^{n+1}$ can be updated thanks to (3.41). Next, in order to enforce the validity of the generalized jump conditions (3.7) at the discrete level, we propose as in the explicit in time procedure to define $(S_i)_K^{n+1}$ from a relevant definition of $(X_i)_K^{n+1}$. More precisely, we first propose the following natural implicit version of (3.45) :

$$\begin{aligned}(X_i)_K^{n+1}-(X_i)_K^n + \frac{\Delta t}{|K|}\sum_{e\in\partial K}(\mathbf{w}\cdot\mathbf{n}_{e,K})_e^n\big((X_i)_{e,K}^{n+1}-(X_i)_K^{n+1}\big)|e| \\ = C_i(\epsilon_K^{n+1}-\epsilon_K^{n+1-}) \quad i=1,...,N-1,\end{aligned} \tag{3.97}$$

where by definition $\epsilon_K^{n+1-}=\sum_{i=1}^N \epsilon_i(\rho_K^{n+1-},(S_i)_K^{n+1-})$, and in agreement with (3.65)-(3.66) :

$$(X_i)_{e,K}^{n+1} = \begin{cases} (X_i)_K^{n+1} & \text{if} \quad (\mathbf{w}\cdot\mathbf{n}_{e,K})_e^n \geq 0, \\ (X_i)_{K_e}^{n+1} & \text{otherwise.} \end{cases} \tag{3.98}$$

The linear system (3.97) to be solved for finding each $(X_i)_K^{n+1}$, $i=1,...,N-1$ involves the same linear operator as for the prediction of the entropy values $(S_i)_K^{n+1-}$, which is interesting from the computational cost point of view. Thanks to lemma 3.4 and provided that the total internal energy $(\rho\epsilon)_K^{n+1}$ is positive, we are then able to define the ultimate values $(S_i)_K^{n+1}$ exactly as in the explicit algorithm :

$$(S_i)_K^{n+1} = (S_N)_K^{n+1}\exp\big((X_i)_K^{n+1}\big), \quad 1\leq i\leq N-1,$$

with $(S_N)_K^{n+1}$ such that $\psi\big((S_N)_K^{n+1}\big) = 0$, that is

$$(\rho\epsilon)_K^{n+1} = \sum_{i=1}^{N-1} \rho_K^{n+1}\epsilon_i\Big(\rho_K^{n+1}, (S_N)_K^{n+1}\exp\big((X_i)_K^{n+1}\big)\Big) + \rho_K^{n+1}\epsilon_N\big(\rho_K^{n+1}, (S_N)_K^{n+1}\big).$$

Numerical experiments attest the positivity of $(\rho\epsilon)_K^{n+1}$ in practice.
The unknown vector $\mathbf{u} = \big(\rho, \rho\mathbf{w}, (\rho S_i)_{1\leq i\leq N}\big)^t$ is now completely determined at time t^{n+1}. This completes the description of the strategy.

3.4. Numerical illustration. We investigate the performance of the proposed time-implicit predictor-corrector procedure for the approximation of the 2D Euler limit equations (1.10) over a reentry blunt body. Residuals are evaluated using a nonlinear second order MUSCL technique based on the van Albada limiter (see [27], [29] for instance). We consider $N = 3$ independent polytropic pressure laws with constant adiabatic coefficients γ_i respectively given by $\gamma_1 = 1.4$, $\gamma_2 = 1.3$, $\gamma_3 = 1.2$. The ratios of the viscosity coefficients are constant with $\frac{\mu_2}{\mu_1} = \frac{\mu_3}{\mu_1} = 1$. The free-stream conditions follow from a Mach number set to $M_\infty = 2$ and static pressures $p_{1,\infty} = 10\ Pa$, $p_{2,\infty} = 26\ Pa$, $p_{3,\infty} = 40\ Pa$ and density $\rho = 4.018\ 10^{-4}\ kgm^{-3}$. They are responsible for the existence of a strong bow shock in the steady state solution. The computational domain consists of a curvilinear mesh made of 32×88 cells, displayed in Fig. (3.1). Fig. (3.2) shows the density contours while Fig. (3.3) depicts the pressure distributions along the stagnation line.

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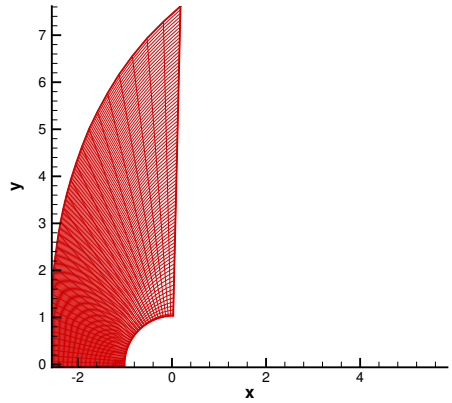

FIG. 3.1. *Mesh*

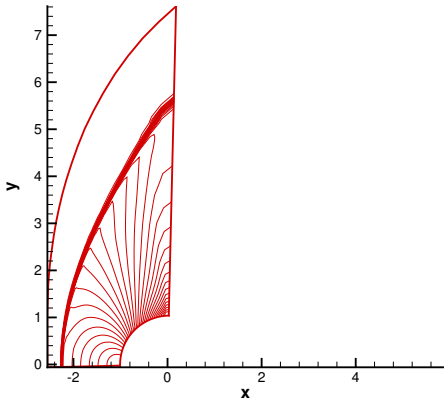

FIG. 3.2. *Density contours*

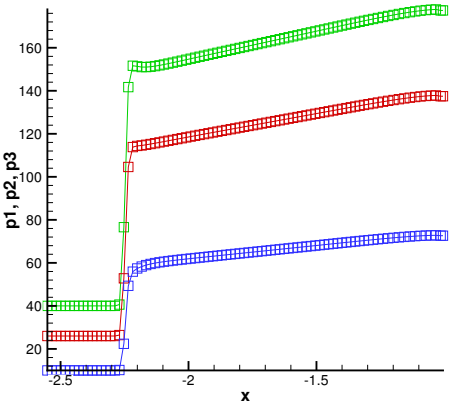

FIG. 3.3. *Pressures on the stagnation line*

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