# Computing phase transitions arising in traffic flow modeling

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**Summary.** A new version of Godunov's scheme is proposed in order to compute the solutions of a traffic flow model with phase transitions. The scheme is based on a modified averaging strategy and a sampling procedure.

## 1 Introduction

We propose a numerical scheme in order to compute the solutions of a traffic flow model with phase transitions. The model has been introduced by Colombo [7] in order to explain empirical flow-density relations. For low densities, the flow is *free* and is described by a scalar conservation law (Lighthill-Whitham [9] and Richards [10] (LWR) model). At high densities the flow is *congested* and is described by a $2 \times 2$ system. We get

$$\begin{array}{ll} \text{Free flow:} & \text{Congested flow:} \\ (\rho, q) \in \Omega_f, & (\rho, q) \in \Omega_c, \\ \partial_t \rho + \partial_x(\rho v) = 0, & \left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t q + \partial_x\left((q - Q)v\right) = 0, \end{array} \right. \\ q = \rho V, & \\ v = v_f(\rho) = V\left(1 - \frac{\rho}{R}\right), & v = v_c(\rho, q) = \left(1 - \frac{\rho}{R}\right)\frac{q}{\rho}. \end{array} \tag{1}$$

The conserved quantity $\rho \in [0, R]$ is the mean traffic density, and $v$ is the mean traffic velocity. The parameter $R$ is the positive maximal density, $V$ the maximal speed and $Q$ is a parameter of the road under consideration. The weighted linear momentum $q$ is originally motivated by gas dynamics. It approximates the real flux $\rho v$ for $\rho$ small compared to $R$.
The coupling is achieved by introducing a transition dynamics from free to congested flow.
The $2 \times 2$ system describing the congested flow turns out to be hyperbolic, the second characteristic field being linearly degenerate, while the first has an

inflection point along the curve $q = Q$. Moreover, shock and rarefaction curves coincide, hence system (1), right, belongs to Temple class [11]. A detailed description of the Riemann solver, and analogies between solutions to (1) and real traffic features are given in [7] (see [8] for the well posedness of the Cauchy problem).
The domain $\Omega_f$ (resp. $\Omega_c$) is taken to be an invariant set for (1), left (resp., right). The resulting domain is given by

$$\Omega_f = \{(\rho, q) \in [0, R] \times [0, +\infty[ : v_f(\rho) \geq V_f,\, q = \rho \cdot V\} ,$$
$$\Omega_c = \left\{(\rho, q) \in [0, R] \times [0, +\infty[ : v_c(\rho, q) \leq V_c,\, \tfrac{q-Q}{\rho} \in \left[\tfrac{Q_- - Q}{R}, \tfrac{Q_+ - Q}{R}\right]\right\} ,$$

where $V_f > V_c$ are the threshold speeds, i.e. above $V_f$ the flow is free and below $V_c$ the flow is congested. The parameters $Q_- \in ]0, Q[$ and $Q_+ \in ]Q, +\infty[$ depend on the environmental conditions and determine the width of the congested region.
The domain $\Omega = \Omega_f \cup \Omega_c$ turns out to be a disconnected set in $\mathbb{R}^2$, its two connected components representing the free and the congested phases. Due to the lack of convexity of the domain, the classical Godunov method is not applicable. In fact, in the presence of phase transitions, the projection step of the algorithm can give values which are not in the domain. Then the procedure is stopped. We are thus led to present a new version of the Godunov scheme, based on a modified averaging strategy and a sampling procedure. More precisely, we modify the mesh cells following the phase boundaries, so that the projection involves only values belonging to the same phase. In order to come back to the original cells, we complete the projection step with a Glimm-type sampling technique. The averaging procedure on modified cells has first been used (up to our knowledge) in [12] but in a different context and a slightly different form. However, the idea of going back to the initial cells by means of a sampling procedure is new and allows us to avoid dealing with moving meshes (as in [12]). Similar numerical techniques have recently been proposed by the first author for approximating nonclassical solutions arising in certain nonlinear hyperbolic equations (see [2], [1] and the references therein), and very recently by Chalons and Coquel in [3] for computing sharp discrete shock profiles.

Of course, the random choice method (Glimm's scheme) could be applied successfully in this case. Nevertheless, our method doesn't need to compute all the values in the Riemann solution, but only the values on both sides of the phase transition, and is then cheaper. Moreover, our algorithm coincides with the classical Godunov scheme, and hence it is conservative, away from phase transitions.

Numerical tests are showed to prove the validity of the method.

## 2 A new version of the Godunov scheme

We will use the following shorten form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0\,, \qquad \mathbf{u} \in \Omega = \Omega_f \cup \Omega_c\,, \tag{2}$$

for the model (1), where

$$\begin{cases} \mathbf{u} = (\rho, q) & \text{and} \quad \mathbf{f}(\mathbf{u}) = (\rho v_f(\rho), q v_f(\rho))\,, & \text{if } (\rho, q) \in \Omega_f\,, \\ \mathbf{u} = (\rho, q) & \text{and} \quad \mathbf{f}(\mathbf{u}) = (\rho v_c(\rho, q), (q - Q) v_c(\rho, q))\,, & \text{if } (\rho, q) \in \Omega_c. \end{cases}$$

From now on, (2) will be supplemented with an initial datum, setting

$$\mathbf{u}(., t = 0) = \mathbf{u}_0 \in \Omega. \tag{3}$$

We introduce a space step $\Delta x$ and a time step $\Delta t$, both assumed to be constant for simplicity. We set $\nu = \Delta t / \Delta x$. Then, we define the mesh interfaces $x_{j+1/2} = j\Delta x$ for $j \in \mathbb{Z}$ and the intermediate times $t^n = n\Delta t$ for $n \in \mathbb{N}$, and we seek at each time $t^n$ an approximation $\mathbf{u}_j^n$ of the solution of (2)-(3) on the interval $[x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$. Therefore, a piecewise constant approximated solution $x \to \mathbf{u}_\nu(x, t^n)$ of the solution $\mathbf{u}$ is given by

$$\mathbf{u}_\nu(x, t^n) = \mathbf{u}_j^n \ \text{ for all } \ x \in C_j = [x_{j-1/2}; x_{j+1/2}), \ \ j \in \mathbb{Z}, \ \ n \in \mathbb{N}.$$

When $n = 0$, we set $x_j = 0.5 \cdot (x_{j-1/2} + x_{j+1/2})$ and $\mathbf{u}_j^0 = \mathbf{u}_0(x_j)$, for all $j \in \mathbb{Z}$. Note that the usual $L^2$-projection is not adapted in the present context since, depending on the proposed initial data, it could artificially introduce unphysical states which are not in the phase space at time $t = 0$ (recall that $\Omega = \Omega_f \cup \Omega_c$ is not convex).

Like the classical Godunov scheme, our method is composed of two steps : a first step in which the solution evolves in time according to the PDE model under consideration, and a second step of projection onto piecewise constant functions.

*Step 1 : Evolution in time*
In this first step, one solves the following Cauchy problem

$$\begin{cases} \partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0, \ \ x \in \mathbb{R}, \\ \mathbf{v}(x, 0) = \mathbf{u}_\nu(x, t^n), \end{cases} \tag{4}$$

for times $t \in [0, \Delta t]$. Recall that $x \to \mathbf{u}_\nu(x, t^n)$ is piecewise constant. Then, under the usual CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{v}} \{|\lambda_i(\mathbf{v})|, \ i = 1 \text{ if } \mathbf{v} \in \Omega_f, \, i = 1, 2 \text{ if } \mathbf{v} \in \Omega_c\} \leq \frac{1}{2}, \tag{5}$$

for all the $\mathbf{v}$ under consideration, the solution of (4) is known by gluing together the solutions of the Riemann problems set at each interface :

$$\mathbf{v}(x, t) = \mathbf{v}_\mathbf{r}(\frac{x - x_{j+1/2}}{t}; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n) \ \text{ for all } \ (x, t) \in [x_j, x_{j+1}] \times [0, \Delta t], \tag{6}$$

where $(x,t) \to \mathbf{v_r}(\frac{x}{t};\mathbf{v}^l,\mathbf{v}^r)$ denotes the self-similar solution of the Riemann problem

$$\begin{cases} \partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0, \;\; x \in \mathbb{R}, \;\; t \in \mathbb{R}^{+,\star} \\ \mathbf{v}(x,0) = \begin{cases} \mathbf{v}^l \text{ if } x < 0, \\ \mathbf{v}^r \text{ if } x > 0, \end{cases} \end{cases}$$

whatever $\mathbf{v}^l$ and $\mathbf{v}^r$ are in the phase space $\Omega_f \cup \Omega_c$.

*Step 2 (Modified) : Projection ($t^n \to t^{n+1}$)*
In the usual Godunov method and to get a piecewise constant approximated solution on each cell $\mathcal{C}_j$ at time $t^{n+1}$, the solution $x \to \mathbf{v}(x,\Delta t)$ given by (6) is simply averaged, as expressed by the following update formula:

$$\mathbf{u}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x,\Delta t) dt, \;\; j \in \mathbb{Z}. \tag{7}$$

In the present context, this strategy may fail due to the lack of convexity of the domain $\Omega_f \cup \Omega_c$ in the $(\rho,q)$-plane and to the possible presence of phase transitions in the Riemann solutions. In this case, the state $\mathbf{u}_j^{n+1}$ resulting from the averaging procedure (7) can be outside $\Omega_f \cup \Omega_c$ for some $j \in \mathbb{Z}$ (even if the solution $\mathbf{u}_\nu(.,t^n)$ belongs to the domain), so that the classical Godunov method stops. We then propose to average the solution $x \to \mathbf{v}(x,\Delta t)$ on (possibly) modified and non uniform cells $\overline{\mathcal{C}}_j^n = [\overline{x}_{j-1/2}^n, \overline{x}_{j+1/2}^n)$ constructed as follows. Let $(\sigma_{j+1/2}^n = \sigma(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n))_{j\in\mathbb{Z}}$ be a sequence of characteristic speeds of propagation at interfaces $(x_{j+1/2})_{j\in\mathbb{Z}}$ such that:

- if $\mathbf{u}_j^n$ and $\mathbf{u}_{j+1}^n$ are not in the same phase (free or congested), then $\sigma_{j+1/2}^n$ coincides with the speed of propagation of the phase transition in the Riemann solution $(x,t) \to \mathbf{v_r}(\frac{x}{t};\mathbf{u}_j^n,\mathbf{u}_{j+1}^n)$;
- if $\mathbf{u}_j^n$ and $\mathbf{u}_{j+1}^n$ belong to the same phase, then $\sigma_{j+1/2}^n = 0$.

Then we define the new interfaces $\overline{x}_{j+1/2}^n$ at time $t^{n+1}$ setting

$$\overline{x}_{j+1/2}^n = x_{j+1/2} + \sigma_{j+1/2}^n \, \Delta t, \quad j \in \mathbb{Z}. \tag{8}$$

We also introduce

$$\overline{\Delta x}_j^n = \overline{x}_{j+1/2}^n - \overline{x}_{j-1/2}^n, \quad j \in \mathbb{Z}.$$

In particular, on each modified cell $\overline{\mathcal{C}}_j^n = [\overline{x}_{j-1/2}^n, \overline{x}_{j+1/2}^n)$, the solution $x \to \mathbf{v}(x,\Delta t)$ given by (6) is fully either in the free phase or in the congested phase. Then, averaging this solution on cells $\overline{\mathcal{C}}_j^n$ provide us with a piecewise constant approximated solution $\overline{\mathbf{u}}_\nu(x,t^{n+1})$ on a non uniform mesh defined by

$$\overline{\mathbf{u}}_\nu(x,t^{n+1}) = \overline{\mathbf{u}}_j^{n+1} \;\text{ for all }\; x \in \overline{C}_j^n, \;\; j \in \mathbb{Z}, \;\; n \in \mathbb{N},$$

with

$$\overline{\mathbf{u}}_j^{n+1} = \frac{1}{\overline{\Delta x}_j^n} \int_{\overline{x}_{j-1/2}^n}^{\overline{x}_{j+1/2}^n} \mathbf{v}(x,\Delta t) dt, \;\; j \in \mathbb{Z}.$$

Let us underline that by the definition of the modified cells, we actually know to which phase each constant state of the solution $\overline{\mathbf{u}}_\nu(x, t^{n+1})$ belongs. In fact, note that both $\Omega_f$ and $\Omega_c$ are convex domains and then are stable under the process of an $L^2$-projection.

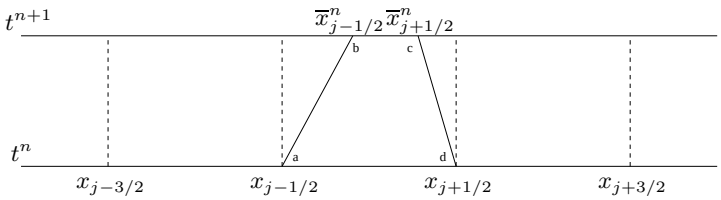


**Fig. 1.** An example of averaging element in the modified Godunov method

We can apply Green's formula on the domain $\overline{E} = (abcd)$ defined by :

$$\overline{E} = \{(x,t):\ t \in [0, \Delta t],\ x_{j-1/2} + \sigma^n_{j-1/2}\, t \le x \le x_{j+1/2} + \sigma^n_{j+1/2}\, t\}$$

(see Figure 1). We get

$$\overline{\mathbf{u}}^{n+1}_j = \frac{\Delta x}{\overline{\Delta x}^n_j}\mathbf{u}^n_j - \frac{\Delta t}{\overline{\Delta x}^n_j}(\overline{\mathbf{f}}^{n,-}_{j+1/2} - \overline{\mathbf{f}}^{n,+}_{j-1/2}) \ \text{ for all } \ j \in \mathbb{Z}, \tag{9}$$

where the numerical fluxes are defined by

$$\overline{\mathbf{f}}^{n,\pm}_{j+1/2} = \mathbf{f}(\mathbf{v_r}(\sigma^{n,\pm}_{j+1/2}; \mathbf{u}^n_j, \mathbf{u}^n_{j+1})) - \sigma^n_{j+1/2}\mathbf{v_r}(\sigma^{n,\pm}_{j+1/2}; \mathbf{u}^n_j, \mathbf{u}^n_{j+1}) \ \text{ for all } \ j \in \mathbb{Z}, \tag{10}$$

using classical notations for the traces of the Riemann solutions at given points.

In order to go back to the (uniform) cells $\mathcal{C}_j$, $j \in \mathbb{Z}$, we now propose to pick up randomly on the cell $\mathcal{C}_j$ a value between $\overline{\mathbf{u}}^{n+1}_{j-1}$, $\overline{\mathbf{u}}^{n+1}_j$ and $\overline{\mathbf{u}}^{n+1}_{j+1}$, in agreement with their rate of presence in the cell. More precisely, given a well distributed random sequence $(a_n)$ within interval $(0,1)$, it amounts to set :

$$\mathbf{u}^{n+1}_j = \begin{cases} \overline{\mathbf{u}}^{n+1}_{j-1} \text{ if } \ a_{n+1} \in (0, \frac{\Delta t}{\Delta x}\max(\sigma^n_{j-1/2}, 0)), \\ \overline{\mathbf{u}}^{n+1}_{j} \text{ if } \ a_{n+1} \in [\frac{\Delta t}{\Delta x}\max(\sigma^n_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x}\min(\sigma^n_{j+1/2}, 0)), \\ \overline{\mathbf{u}}^{n+1}_{j+1} \text{ if } \ a_{n+1} \in [1 + \frac{\Delta t}{\Delta x}\min(\sigma^n_{j+1/2}, 0), 1), \end{cases} \tag{11}$$

for all $j \in \mathbb{Z}$. Following Collela [6], we consider the van der Corput random sequence $(a_n)$ defined by $a_n = \sum_{k=0}^m i_k 2^{-(k+1)}$, where $n = \sum_{k=0}^m i_k 2^k$, $i_k = 0, 1$, denotes the binary expansion of the integers $n = 1, 2, ....$ This well-known sequence is often favorite since, when used in the context of Glimm's scheme, it leads to very good results in the smooth parts of the solutions (see for instance [6] and [5] for some illustrations).

We now propose to test our algorithm on three Riemann problems leading to solutions involving phase transitions. The parameters of the model are taken

to be $R = 1,\ V = 2,\ V_f = 1,\ V_c = 0.85,\ Q = 0.5,\ Q^- = 0.25,\ Q^+ = 1.5$. The numerical solutions will be represented by the density and velocity profiles, and will be compared to the exact solutions. Solutions computed by means of a second-order extension of the method (see [4] for a full description) are also proposed.

For **Test A**, we consider $\rho^l = 0.7,\ \rho^l v^l = 0.3$ in the congested phase and $\rho^r = 0.3$ in the free phase, leading to a solution made of a rarefaction in the congested phase, followed by a phase transition to a free state, itself followed by a rarefaction wave in the free phase. The solutions are plotted on Figure 2 at time $T_f = 0.5$. For this test case, we have used a mesh containing 500 points ($\Delta x = 0.002$).

We now address the case of phase transitions from a free state to a congested state. For **Test B**, we choose $\rho^l = 0.35$ in the free phase and $\rho^r = 0.6$, $\rho^r v^r = 0.25$, in the congested phase. The corresponding solution is a shock-like phase transition followed by a contact discontinuity. Figure 3 plots the solution at time $T_f = 0.6$ with $\Delta x = 0.002$.

For **Test C**, we take $\rho^l = 0.215$ in the free phase and $\rho^r = 0.7$, $\rho^r v^r = 0.2$, in the congested phase, leading a solution composed of a phase transition followed by a rarefaction wave, and a contact discontinuity propagating with a positive speed. In this case the congested state of the phase transition is very difficult to capture properly, due to the numerical diffusion of the scheme which is present in the rarefaction wave. Note that this state is always over-estimated from the proposed averaging strategy. However we observe a good agreement between the numerical solution and the exact solution, and the numerical solution becomes better when the order of accuracy of the method is higher, as it is illustrated on Figure 4 where we have taken $\Delta x = 0.01$, $\Delta x = 0.002$ and $\Delta x = 0.001$, $T_f = 0.8$.

## 3 Conservation error

Due to the random sampling present in *Step 2 (Modified)*, our method does not strictly conserve the mass $\rho$. We then propose to measure the conservation errors on piecewise constant numerical solution $\rho_\nu$ defined as

$$\rho_\nu(x,t) = \rho_j^n \quad \text{if} \quad (x,t) \in [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1}),$$

between times $t = 0$ and $t = T$, for some $T > 0$. We denote $[x_0, x_1]$ the computational domain and we proceed exactly as in [2] : we compare with 0 the function $E : T \in \mathbb{R}^+ \to E(T) \in \mathbb{R}$ with $E(T)$ defined by

$$\int_{x_0}^{x_1} \rho_\nu(x,T)dx \times E(T) = \int_{x_0}^{x_1} \rho_\nu(x,T)dx - \int_{x_0}^{x_1} \rho_\nu(x,0)dx + \int_0^T \{\rho v_c(\rho,q)\}_\nu(x_1,t)dt - \int_0^T \{\rho v_c(\rho,q)\}_\nu(x_0,t)dt. \tag{12}$$

Recall that $q = \rho V$ in the free phase. $E(T)$ represents the relative conservation error of $\rho$ at time $T$ on the interval $[x_0, x_1]$. In the next table, we give for the **Tests A, B, C** the values of the $L^1$-norm $\frac{1}{T_f}||E||_{L^1(0,T_f)}$ of $E$, namely

$$\frac{1}{T_f}||E||_{L^1(0,T_f)} = \frac{1}{T_f}\int_0^{T_f} |E(T)|dT = \sum_{t^n=0}^{t^{n+1}=T_f} \frac{(t^{n+1}-t^n)}{T_f}|E(t^n)|,$$

where $T_f$ is the final time of the corresponding simulations. We observe that the conservation errors are very small and decrease with the mesh size.

| # of points | **Test A** ($x_0 = -0.5$, $x_1 = 0.5$) | **Test B** ($x_0 = -0.3$, $x_1 = 0.3$) | **Test C** ($x_0 = -0.35$, $x_1 = 0.25$) |
|---|---|---|---|
| 50 | 1.58% | 2.02% | 2.23% |
| 100 | 0.81% | 1.01% | 1.11% |
| 500 | 0.18% | 0.17% | 0.22% |

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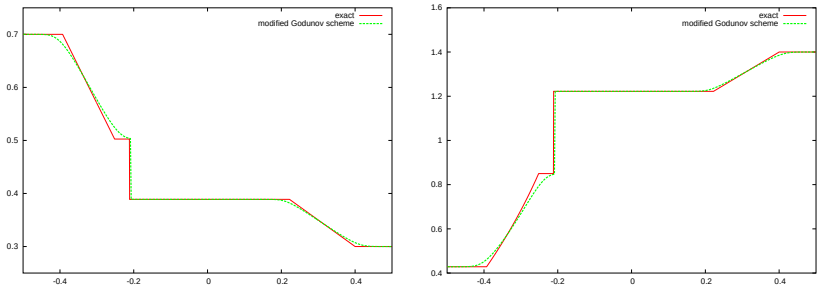


**Fig. 2. Test A** : $\rho$ (Left) and $v$ (Right)

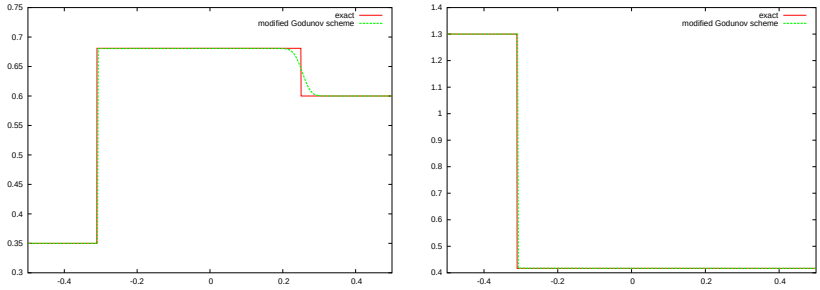


**Fig. 3. Test B** : $\rho$ (Left) and $v$ (Right)

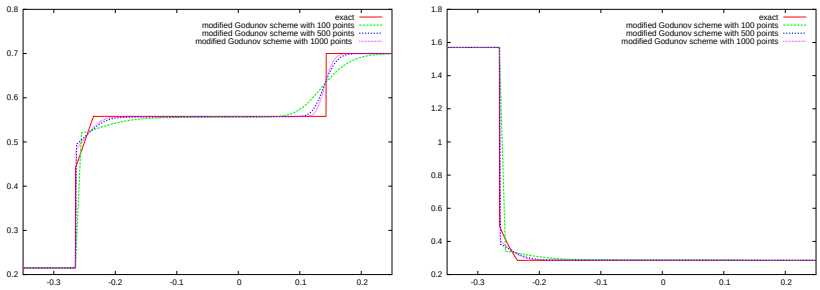


**Fig. 4. Test C** : $\rho$ (Left) and $v$ (Right)