Homogeneous models with phase transition: coupling by finite volume methods

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ABSTRACT. *We study two separate domains sharing a fixed interface. In each one, a different hyperbolic model is used to describe the flow. We propose appropriate conditions at the interface in a way to obtain a coherent description of the unsteady flow according to physical considerations. The problem we consider is the coupling of the homogeneous equilibrium model and the homogeneous relaxation model. Several coupling conditions are described and illustrated by numerical results.*

RÉSUMÉ. *On étudie le problème modèle du couplage de deux systèmes hyperboliques séparés par une interface fixe, les deux systèmes étant différents. Nous proposons des conditions de couplage pour obtenir des écoulements instationnaires cohérents avec des principes physiques. Les modèles considérés ici sont, d'un côté, le modèle homogène à l'équilibre, et, de l'autre côté, le modèle homogène de relaxation. Différentes stratégies de couplage sont proposées et comparées numériquement.*

KEYWORDS: *Model coupling, hyperbolic system, phase transition.*

MOTS-CLÉS : *Couplage de modèles, système hyperbolique, transition de phase.*

1. Introduction

The model coupling arises in the frame of the Neptune project [HER 05a], which involves several numerical and physical problems. One of these problems is the numerical coupling of two-phase flow codes which are based on different models and different numerical methods. These models are different in order to take into account

the different configurations of a global flow. The main difficulty occurs when two different models are put side to side: the information to be transmitted from one model to the other has to be defined. Moreover, even if some physical requirements lead to a coherent description of the coupling, the numerical aspect of coupling must be handled with care. Indeed, in some very simple cases, the numerical coupling method cannot be in agreement with physical principles.

We investigate here the numerical coupling of two different first order systems of partial differential equations. These problems model the flow of a fluid which can be under its vapor form or under its liquid form. We consider models based on the conservation of mass, momentum and total energy, and the mechanical equilibrium (*ie* $p_1 = p_2$) is assumed to be fulfilled. The coupling is located at a fixed interface. On one side, we assume that the phase transition is instantaneous and on the other side, the thermodynamical equilibrium is reached with a non-zero relaxation time. We present several ways of coupling the two systems, from the mathematical and numerical points of view, following some previous works [GOD 04], [GOD 05], [AMB 05]. At the end, we present numerical tests.

2. The homogeneous models

We first present the two models we aim at coupling. The fluid we consider can be in its "vapor" phase or in its "liquid" phase and we allow phase transitions. The first model, the homogeneous equilibrium model, assumes that the thermodynamical equilibrium is fulfilled instantaneously whereas the second model, the homogeneous relaxation model, is a relaxed version, with respect to the difference of the chemical potentials, of the equilibrium model.

2.1. *The homogeneous equilibrium model*

The first model we consider is the homogeneous equilibrium model (HEM):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_L) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p_L)) = 0, \end{cases} \tag{1}$$

where $E = u^2/2 + \varepsilon$ and $p_L = p_L(\rho, \varepsilon)$. The notations are classical: ρ is the density, u the velocity, E the total energy, p_L the pressure and ε the internal energy. We assume that the fluid is a composed by two immiscible perfect gas type phases, with different adiabatic coefficients $\gamma_1 > 1$ and $\gamma_2 > 1$. The pressure law is given by

$$p_L(\rho, \varepsilon) = \begin{cases} (\gamma_1 - 1)\rho\varepsilon, & \text{if } \rho \leq \rho_1^*, \\ (\gamma_1 - 1)\rho_1^*\varepsilon, & \text{if } \rho_1^* < \rho < \rho_2^*, \\ (\gamma_2 - 1)\rho\varepsilon, & \text{if } \rho_2^* \leq \rho, \end{cases} \tag{2}$$

where ρ_1^* and ρ_2^* are two constants defined by

$$\rho_1^* = A\left(\frac{\gamma_1-1}{\gamma_2-1}\right)^{\frac{\gamma_2-1}{\gamma_1-\gamma_2}} \quad \text{and} \quad \rho_2^* = A\left(\frac{\gamma_1-1}{\gamma_2-1}\right)^{\frac{\gamma_1-1}{\gamma_1-\gamma_2}},$$

with

$$A = \exp\left(\frac{\gamma_1 + (\gamma_1-1)\ln(\gamma_1-1) - \gamma_2 - (\gamma_2-1)\ln(\gamma_2-1)}{\gamma_2-\gamma_1}\right).$$

See [CAR 04] and [JAO 01] for more details.

The equation of state (2) is very simple and it takes into account the phase transition: if $\rho \in (0, \rho_1^*]$ the fluid is in its "vapor" form, if $\rho \in [\rho_2^*, \infty)$, the fluid is in its "liquid" form and if $\rho \in (\rho_1^*, \rho_2^*)$, it corresponds to a mixture.

In the following, we will adopt the condensed notation of (1):

$$\partial_t \mathbf{U}_L + \partial_x \mathbf{F}_L(\mathbf{U}_L) = 0, \tag{3}$$

with $\mathbf{U}_L = {}^t(\rho, \rho u, \rho E)$ and $\mathbf{F}_L(\mathbf{U}_L) = {}^t(\rho u, \rho u^2 + p_L, u(\rho E + p_L))$. We also define $\Omega_L = \{\mathbf{U}_L \in \mathbb{R}^3, \rho > 0, \varepsilon > 0\}$ and note that (1)-(2) is strictly hyberbolic over this natural phace space.

2.2. *The homogeneous relaxation model*

We focus now on the homogeneous relaxation model (HRM):

$$\begin{cases} \partial_t m_1 + \partial_x(m_1 u) = \lambda(m_1^*(\rho) - m_1), \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_R) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p_R)) = 0, \end{cases} \tag{4}$$

where λ is a positive constant and m_1 is the partial density of the "vapor". The pressure law is now

$$p_R(\rho, \varepsilon, m_1) = ((\gamma_1 - 1)m_1 + (\gamma_2 - 1)(\rho - m_1))\varepsilon. \tag{5}$$

The function m_1^* is defined by

$$m_1^*(\rho) = \begin{cases} \rho, & \text{if } \rho \le \rho_1^*, \\ \rho_1^* \frac{\rho - \rho_2^*}{\rho_1^* - \rho_2^*}, & \text{if } \rho_1^* < \rho < \rho_2^*, \\ 0, & \text{if } \rho_2^* \le \rho. \end{cases} \tag{6}$$

Let us emphasize that the adiabatic coefficients γ_1 and γ_2 are the same as the adiabatic coefficients of the HEM. Therefore, one can check that when $\lambda \to +\infty$ (that is when $m_1 \to m_1^*$), the HRM formally tends to the HEM. In other words, we have

$$p_R(\rho, \varepsilon, m_1^*(\rho)) = p_L(\rho, \varepsilon), \quad \forall \rho, \varepsilon > 0.$$

Here again, we introduce a short notation for (4):

$$\partial_t \mathbf{U}_R + \partial_x \mathbf{F}_R(\mathbf{U}_R) = \mathbf{S}_R(\mathbf{U}_R), \tag{7}$$

with $\mathbf{U}_R = {}^t(m_1, \rho, \rho u, \rho E)$, $\mathbf{F}_R(\mathbf{U}_R) = {}^t(m_1 u, \rho u, \rho u^2 + p_R, u(\rho E + p_R))$ and $\mathbf{S}_R(\mathbf{U}_R) = {}^t(\lambda(m_1^*(\rho) - m_1), 0, 0, 0)$. The set of admissible states is $\Omega_R = \{\mathbf{U}_R \in \mathbb{R}^4, 0 \leq m1 \leq \rho, \rho > 0, \varepsilon > 0\}$ and the system turns out to be strictly hyperbolic over Ω_R.

3. The mathematical model of the coupled problem

We consider the two different hyperbolic systems presented above separated by the fixed interface $\{x = 0\}$

$$\partial_t \mathbf{U}_L + \partial_x \mathbf{F}_L(\mathbf{U}_L) = 0, \qquad t > 0, x < 0, \tag{8}$$

$$\partial_t \mathbf{U}_R + \partial_x \mathbf{F}_R(\mathbf{U}_R) = \mathbf{S}_R(\mathbf{U}_R), \qquad t > 0, x > 0, \tag{9}$$

with the initial conditions

$$\mathbf{U}_L(x, 0) = \mathbf{U}_{L,0}(x), \quad x < 0, \tag{10}$$

$$\mathbf{U}_R(x, 0) = \mathbf{U}_{R,0}(x), \quad x > 0. \tag{11}$$

In order to couple the two systems (8) and (9), we will introduce two operators

$$\Pi_R^L : \mathbf{U}_L = {}^t(\rho, \rho u, \rho E) \quad \longmapsto \quad \Pi_R^L(\mathbf{U}_L) = {}^t(m_1^*(\rho), \rho, \rho u, \rho E), \tag{12}$$

$$\Pi_L^R : \mathbf{U}_R = {}^t(m_1, \rho, \rho u, \rho E) \quad \longmapsto \quad \Pi_L^R(\mathbf{U}_R) = {}^t(\rho, \rho u, \rho E), \tag{13}$$

which will enable us to convert one variable into the other.

We present three approaches, according to the information which must be transmitted or according to physical requirements: global conservation, continuity of some physical variables through $x = 0$. In the following, the source term $\mathbf{S}_R(\mathbf{U}_R)$ is omitted for simplicity (and without any restriction).

3.1. *The flux coupling*

The first approach of coupling relies on the use of a *color* function $Y : (x, t) \in \mathbb{R} \times (0, \infty) \mapsto [0, 1]$ and it is based on a simple definition of the coupling which provides the global conservation of the unknowns. We replace the systems (8) and (9) by the *augmented* system

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, Y) = 0, \quad x \in \mathbb{R}, t > 0, \\ \partial_t Y = 0, \end{cases} \tag{14}$$

with $Y(x<0,0)=0$ and $Y(x>0,0)=1$. The new variable $\mathbf{U}$ is defined by

$$\mathbf{U}(x,t)=\begin{cases}\Pi_R^L(\mathbf{U}_L)(x,t) & \text{if } x<0,\\ \mathbf{U}_R(x,t) & \text{if } x>0,\end{cases}$$

and the new flux function $\mathbf{F}(\mathbf{U},Y)$ by

$$\mathbf{F}(\mathbf{U},Y)=(1-Y)\mathbf{F}(\mathbf{U},0)+Y\mathbf{F}(\mathbf{U},1)$$

with

$$\mathbf{F}(\mathbf{U},0)={}^t(0,\,{}^t\mathbf{F}_L(\Pi_L^R(\mathbf{U})))\quad\text{and}\quad\mathbf{F}(\mathbf{U},1)=\mathbf{F}_R(\mathbf{U}).$$

This model of coupling is conservative. Nonetheless, the system (14) is resonant, which means that if an eigenvalue of the jacobian matrix $D_{\mathbf{U}}\mathbf{F}(\mathbf{U},Y)$ vanishes, the system admits two null eigenvalues and the eigenvectors do not form a basis of $\mathbb{R}^4$ anymore. In order to avoid this problem, the second equation of (14) can be replaced by

$$\partial_t(\rho Y)+\partial_x(\rho Y u)=\mu\rho(Y_0-Y), \tag{15}$$

where we will make μ tend to $+\infty$ formally.

3.2. *The intermediate state coupling*

The idea of this technique is to impose the continuity of the variables through the interface, namely

$$\Pi_R^L(\mathbf{U}_L)(0_-,t)=\mathbf{U}_R(0_+,t),\quad t>0. \tag{16}$$

Since we are focusing on the coupling of systems that may develop discontinuities, the continuity condition (16) cannot be fulfilled in all cases, in particular when the characteristics of the two problems are incompatible. Therefore, a weak formulation of this condition is proposed:

$$\mathbf{U}_L(0_-,t)\in\mathcal{O}_L(\Pi_L^R(\mathbf{U}_R)(0_+,t)),\ t>0, \tag{17}$$

$$\mathbf{U}_R(0_+,t)\in\mathcal{O}_R(\Pi_R^L(\mathbf{U}_L)(0_-,t)),\ t>0, \tag{18}$$

with

$$\mathcal{O}_L(\mathbf{U}_b)=\{\mathcal{W}_L(0_-;\mathbf{U},\mathbf{U}_b),\mathbf{U}\in\Omega_L\}, \tag{19}$$

$$\mathcal{O}_R(\mathbf{U}_b)=\{\mathcal{W}_R(0_+;\mathbf{U}_b,\mathbf{U}),\mathbf{U}\in\Omega_R\}, \tag{20}$$

where $\mathcal{W}_\alpha(x/t;\mathbf{U}_g,\mathbf{U}_d)$, $\alpha=L,R$, is the self-similar solution of the Riemann problem

$$\begin{cases}\partial_t\mathbf{U}_\alpha+\partial_x\mathbf{F}_\alpha(\mathbf{U}_\alpha)=0,\quad x\in\mathbb{R},t>0,\\ \mathbf{U}_\alpha(x,0)=\begin{cases}\mathbf{U}_g & \text{if } x<0,\\ \mathbf{U}_d & \text{if } x>0.\end{cases}\end{cases}$$

The condition (17) (respectively (18)) means that the state $\Pi_L^R(\mathbf{U}_R)(0_+,t)$ (resp. $\Pi_R^L(\mathbf{U}_L)(0_-,t)$) is an admissible boundary condition (see [DUB 88]) for the system (8) (resp. (9)). More details can be found in [GOD 05].

3.3. *The modified intermediate state coupling*

The conditions (17)-(18) provide, whenever it is possible, the continuity of the variables m_1 (with $m_1 = m_1^*$ for $x < 0$), ρ, ρu and ρE through the interface. Let us assume that ρ, ρu and ρE are continuous at $x = 0$ and that $\rho u(0,t) < 0$. In such a case, if $m_1 \neq m_1^*$ in the right part of the domain, the continuity of m_1 at $x = 0$ cannot be ensured. As a result, the pressure cannot be continuous at $x = 0$:

$$p_L(\rho,\varepsilon)(0_-,t) = p_R(\rho,\varepsilon,m_1^*(\rho))(0_-,t) \neq p_R(\rho,\varepsilon,m_1)(0_+,t).$$

But, for physical considerations, one could prefer the solution to fulfill the continuity of the pressure at $x = 0$.

Therefore, instead of (17)-(18), the coupling condition should be a weakened form of

$$\phi_L(\Pi_R^L(\mathbf{U}_L))(0_-,t) = \phi_R(\mathbf{U}_R(0_+,t)), \quad t > 0, \tag{21}$$

where ϕ_L and ϕ_R are two changes of variable from Ω_R to Ω_R. In our case, they are

$$\phi_L(m_1,\rho,\rho u,\rho E) = (m_1,\rho,\rho u, p_L(\rho,\varepsilon)),$$
$$\phi_R(m_1,\rho,\rho u,\rho E) = (m_1,\rho,\rho u, p_R(\rho,\varepsilon,m_1)).$$

The weak coupling conditions associated with (21) are

$$\mathbf{U}_L(0_-,t) \in \mathcal{O}_L(\Pi_L^R(\phi_L^{-1}(\phi_R(\mathbf{U}_R)))(0_+,t)),\ t > 0, \tag{22}$$

$$\mathbf{U}_R(0_+,t) \in \mathcal{O}_R(\phi_R^{-1}(\phi_L(\Pi_R^L(\mathbf{U}_L)))(0_-,t)),\ t > 0, \tag{23}$$

using the previous definitions (19)-(20) of $\mathcal{O}_L$ and $\mathcal{O}_R$.

4. Numerical coupling of hyperbolic systems

We focus now on the simulation of the coupling problem (8)-(9)-(10)-(11). Of course, the numerical coupling must vary according to the type of coupling we want to impose.

Let Δt and Δx be the time and the space steps. The interfaces of the mesh are defined as $x_j = j\Delta x$, $j \in \mathbb{Z}$, and the intermediate times are $t^n = n\Delta t$, $n \in \mathbb{N}$. The classical finite volume approximation is used:

$$\Delta x(\mathbf{U}_\alpha)_{j+1/2}^0 = \int_{x_j}^{x_{j+1}} \mathbf{U}_{\alpha,0}(x)\,dx, \quad \alpha = L, R.$$

The approximation $(\mathbf{U}_{\alpha,j+1/2}^n)_{j,n}$ is given by the numerical scheme

$$\mathbf{U}_{L,j+1/2}^{n+1} = \mathbf{U}_{L,j+1/2}^n - \frac{\Delta t}{\Delta x}(\mathbf{G}_{L,j+1}^n - \mathbf{G}_{L,j}^n), \quad \text{for } j < 0, n \geq 0, \tag{24}$$

$$\mathbf{U}_{R,j+1/2}^{n+1} = \mathbf{U}_{R,j+1/2}^n - \frac{\Delta t}{\Delta x}(\mathbf{G}_{R,j+1}^n - \mathbf{G}_{R,j}^n), \quad \text{for } j \geq 0, n \geq 0, \tag{25}$$

where the numerical fluxes are defined with the help of two consistant functions $\mathbf{G}_L(\mathbf{U}_L, \mathbf{V}_L)$ and $\mathbf{G}_R(\mathbf{U}_R, \mathbf{V}_R)$ with the fluxes $\mathbf{F}_L(\mathbf{U}_L)$ and $\mathbf{F}_R(\mathbf{U}_R)$:

$$\mathbf{G}^n_{L,j} = \mathbf{G}_L(\mathbf{U}^n_{L,j-1/2}, \mathbf{U}^n_{L,j+1/2}), \quad j < 0, n \geq 0, \tag{26}$$

$$\mathbf{G}^n_{R,j} = \mathbf{G}_R(\mathbf{U}^n_{R,j-1/2}, \mathbf{U}^n_{R,j+1/2}), \quad j > 0, n \geq 0. \tag{27}$$

Note that it remains to define the numerical fluxes $\mathbf{G}^n_{L,0}$ and $\mathbf{G}^n_{R,0}$ at the coupling interface. We restrict this presentation to three-point schemes for simplicity, though we will use five-point schemes in applications (the extension is obvious).

We present several ways to define these two numerical fluxes and relate their definition to the previous models of coupling.

4.1. *The numerical flux coupling*

The following numerical method is dedicated to the simulation of the flux coupling. The way of defining $\mathbf{G}^n_{L,0}$ and $\mathbf{G}^n_{R,0}$ consists in solving the problem (14) with the initial data

$$\begin{aligned} \mathbf{U}(x,0) &= \begin{cases} \mathbf{U}^n_{-1/2} & \text{if } x < 0, \\ \mathbf{U}^n_{+1/2} & \text{if } x > 0, \end{cases} \\ Y(x,0) &= \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \end{aligned} \tag{28}$$

It has been investigated in [HER 05b] and it gives precise results. But, in order to avoid any resonance phenomenon, we use the equation (15) instead of $\partial_t Y = 0$ in (14). This system is solved in two steps. First, the convective part is solved, that is to say

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, Y) = 0, \\ \partial_t(\rho Y) + \partial_x(\rho Y u) = 0, \end{cases}$$

with the initial data (28). It gives an exact (or approximate) value of $(\mathbf{U}^*, Y^*)(x/t)$ and we set

$$\mathbf{G}^n_{L,0} = \mathbf{G}^n_{R,0} = \mathbf{F}(\mathbf{U}^*(0), Y^*(0)). \tag{29}$$

In the second step, the equilibrium is recovered: $Y^* = Y_0$. As a result, this method is completely conservative. We refer to [AMB 05] for more details.

4.2. *The two-flux method*

This numerical method is based on the introduction of two reconstructed states $\overline{\mathbf{U}}^n_{R,-1/2}$ and $\overline{\mathbf{U}}^n_{L,1/2}$. The numerical fluxes at $x = 0$ are given by

$$\mathbf{G}^n_{L,0} = \mathbf{G}_L(\mathbf{U}^n_{L,-1/2}, \overline{\mathbf{U}}^n_{L,1/2}), \tag{30}$$

$$\mathbf{G}^n_{R,0} = \mathbf{G}_R(\overline{\mathbf{U}}^n_{R,-1/2}, \mathbf{U}^n_{R,1/2}). \tag{31}$$

Such a numerical coupling aims at approximating the (modified or not) intermediate state coupling. If the variable to be transmitted at the interface is the conservative variable, we use

$$\overline{\mathbf{U}}^n_{R,-1/2} = \Pi^L_R(\mathbf{U}^n_{L,-1/2}) \quad \text{and} \quad \overline{\mathbf{U}}^n_{L,1/2} = \Pi^R_L(\mathbf{U}^n_{R,1/2}).$$

For what concerns the modified intermediate state coupling, we use

$$\overline{\mathbf{U}}^n_{R,-1/2} = \phi_R^{-1}(\phi_L(\Pi^L_R(\mathbf{U}^n_{L,-1/2}))) \quad \text{and} \quad \overline{\mathbf{U}}^n_{L,1/2} = \Pi^R_L(\phi_L^{-1}(\phi_R(\mathbf{U}^n_{R,1/2}))).$$

For more details, see [GOD 04] and [GOD 05].

4.3. *The numerical schemes*

We describe in this section the numerical schemes we use in each domain. The same numerical scheme has been used on each side, in order to simplify the comparison. We present it in the frame of the HRM, which includes the frame of the HEM (we have dropped the subscripts L and R). The scheme we use is a Lagrange-Projection scheme (see for instance [DEP 01]). Actually, it is a five-point scheme, that is to say the flux $\mathbf{G}^n_j$ depends on four states:

$$\mathbf{G}^n_j = \mathbf{G}(\mathbf{U}^n_{j-3/2}, \mathbf{U}^n_{j-1/2}, \mathbf{U}^n_{j+1/2}, \mathbf{U}^n_{j+3/2}, \Delta t/\Delta x).$$

Note that it also depends on the ratio $\Delta t/\Delta x$. This scheme is based on two steps.

The *Lagrange step* is solved with the help of the acoustic scheme:

$$\begin{aligned}
y^{n+1-}_{j+1/2} &= y^n_{j+1/2},\\
\tau^{n+1-}_{j+1/2} &= \tau^n_{j+1/2} + \frac{\Delta t}{\rho^n_{j+1/2}\Delta x}(u^n_{j+1} - u^n_j),\\
u^{n+1-}_{j+1/2} &= u^n_{j+1/2} - \frac{\Delta t}{\rho^n_{j+1/2}\Delta x}(p^n_{j+1} - p^n_j),\\
E^{n+1-}_{j+1/2} &= E^n_{j+1/2} - \frac{\Delta t}{\rho^n_{j+1/2}\Delta x}((pu)^n_{j+1} - (pu)^n_j),
\end{aligned}$$

with

$$\begin{aligned}
u^n_j &= (u^n_{j-1/2} + u^n_{j+1/2})/2 + (p^n_{j-1/2} - p^n_{j+1/2})/(2(\rho c)^n_j),\\
p^n_j &= (p^n_{j-1/2} + p^n_{j+1/2})/2 + (\rho c)^n_j(u^n_{j-1/2} - u^n_{j+1/2})/2,
\end{aligned}$$

where $y = m_1/\rho$, $\tau = 1/\rho$ and $(\rho c)^n_j = \max((\rho c)^n_{j-1/2}, (\rho c)^n_{j+1/2})$. We then obtain $(\mathbf{U}^{n+1-}_{j+1/2})_j$. The grid points have moved at the fluid velocity, approximated here by u^n_j. The quantities $(\mathbf{U}^{n+1-}_{j+1/2})_j$ are then defined on a Lagrangian grid $(x^*_j)_j$ defined by $x^*_j = x_j + u^n_j\Delta t$.

The second step is the *projection step*. We define $\mathbf{U}^{n+1-}(x)$ as a function which is constant on each cell of the Lagrangian grid $(x_j^*)_j$ and given by $\mathbf{U}_{j+1/2}^{n+1-}$ in $[x_j^*; x_{j+1}^*)$ for all $j \in \mathbb{Z}$. We then project $\mathbf{U}^{n+1-}$ on the Eulerian grid $(x_j)_j$: $\Delta x \mathbf{U}_{j+1/2}^{n+1} = \int_{x_j}^{x_{j+1}} \mathbf{U}^{n+1-}(x)\, dx$.

As a result, we get for $\mathbf{U}_{j+1/2}^{n+1}$:

$$
\begin{aligned}
(m_1)_{j+1/2}^{n+1} &= (m_1)_{j+1/2}^{n} - \frac{\Delta t}{\Delta x}(u_{j+1}^n (m_1)_{\beta_{j+1,n}}^{n+1-} - u_j^n (m_1)_{\beta_{j,n}}^{n+1-}),\\
\rho_{j+1/2}^{n+1} &= \rho_{j+1/2}^{n} - \frac{\Delta t}{\Delta x}(u_{j+1}^n \rho_{\beta_{j+1,n}}^{n+1-} - u_j^n \rho_{\beta_{j,n}}^{n+1-}),\\
(\rho u)_{j+1/2}^{n+1} &= (\rho u)_{j+1/2}^{n} - \frac{\Delta t}{\Delta x}(u_{j+1}^n (\rho u)_{\beta_{j+1,n}}^{n+1-} - u_j^n (\rho u)_{\beta_{j,n}}^{n+1-} + p_{j+1}^n - p_j^n),\\
(\rho E)_{j+1/2}^{n+1} &= (\rho E)_{j+1/2}^{n} - \frac{\Delta t}{\Delta x}(u_{j+1}^n (\rho E)_{\beta_{j+1,n}}^{n+1-} - u_j^n (\rho E)_{\beta_{j,n}}^{n+1-}\\
&\qquad + (pu)_{j+1}^n - (pu)_j^n),
\end{aligned}
$$

where

$$
\beta_{j,n} = \begin{cases} j - 1/2 & \text{if } u_j^n > 0,\\ j + 1/2 & \text{if } u_j^n < 0. \end{cases}
$$

This scheme is entropy consistent and positive (for the density ρ, the partial density m_1 and the internal energy ε) under a classical CFL condition [DEP 01].

4.4. *Numerical tests*

In order to clarify the results, we only focus on the case $\lambda = 0$. We set $\gamma_1 = 1.6$ and $\gamma_2 = 1.4$ (which gives $\rho_1^* \approx 0.613$ and $\rho_2^* \approx 0.919$). The initial condition is

	$x < 0$	$x > 0$
partial density	—	2
density	1	2
velocity	-1/2	-1/2
pressure	1	1

It corresponds to a contact discontinuity going left. On figure 1, “Flux”, “State CV” and “State Pr” respectively denote to the flux coupling, the state coupling with the conservative variable and the modified state coupling with the continuity of pressure. These results have been obtained with 1000 cells and with a Courant number equal to 0.4. We can see that only the modified state coupling enables to preserve the velocity and the pressure constant. Moreover, a mixture ($0 < m_1 < \rho$) appears in the negative part of the domain for the conservative state coupling. Note that, if we aim at preserve the contact discontinuity through the coupling interface, the modified state coupling with the continuity of pressure must be used.

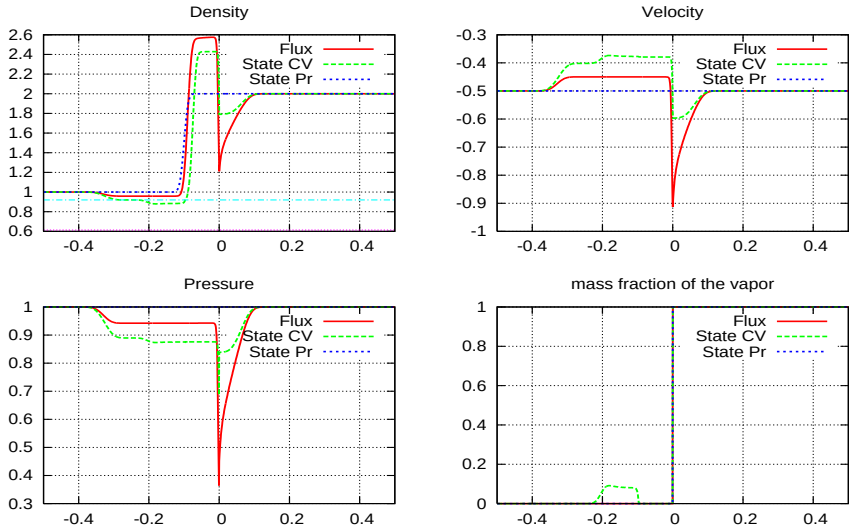

Figure 1. *Comparison of the three methods of coupling for a contact discontinuity.*

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