A Roe-type linearization for the gaz dynamics equations

We focus, as we did before, on the case of a polytropic ideal gas:

$$p = (\gamma - 1)\rho \mathcal{E}, \quad \mathcal{E} = C_v T.$$

We propose to seek for a Roe-type linearization of the form

$$A(u_L, u_R) = \nabla_u f(\bar{u}(u_L, u_R)) \qquad u = (\rho, \rho u, \rho E)$$

where $\bar{u}$ will be such that $\bar{u}(u, u) = u$. Doing so, (ii) and (iii) are automatically satisfied provided that $\bar{u}(u_L, u_R)$ belongs to the phase space. It remains to check the property (i), namely

$$f(u_R) - f(u_L) = A(u_L, u_R)(u_R - u_L).$$

We can easily show that the Jacobian matrix $\nabla_u f$ is given by

$$\nabla f(u) = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & -(\gamma-3)u & (\gamma-1) \\ u\left((\gamma-1)\frac{u^2}{2} - H\right) & H - (\gamma-1)u^2 & \gamma u \end{pmatrix}$$

with $H = h + \frac{u^2}{2}$ and $h = \mathcal{E} + p/\rho = \mathcal{E} + (\gamma-1)\mathcal{E} = \gamma \mathcal{E}$

We then look for $\bar{u}$ and $\bar{H}$ such that

$$\begin{pmatrix} (\rho u)_R - (\rho u)_L \\ (\rho u^2 + p)_R - (\rho u^2 + p)_L \\ (\rho E u + p u)_R - (\rho E u + p u)_L \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{\bar{u}^2}{2} & -(\gamma-3)\bar{u} & (\gamma-1) \\ \bar{u}\left((\gamma-1)\frac{\bar{u}^2}{2} - \bar{H}\right) & \bar{H} - (\gamma-1)\bar{u}^2 & \gamma \bar{u} \end{pmatrix} \begin{pmatrix} \rho_R - \rho_L \\ (\rho u)_R - (\rho u)_L \\ (\rho E)_R - (\rho E)_L \end{pmatrix}$$

The first component is clearly satisfied.

The second one writes

$$[\rho u^2+p] = \frac{\gamma-3}{2}\bar{u}^2[\rho] - (\gamma-3)\bar{u}[\rho u] + (\gamma-1)\left\{\frac{[\rho u^2]}{2} + \frac{[p]}{\gamma-1}\right\}$$

$$= \frac{\gamma-3}{2}\bar{u}^2[\rho] - (\gamma-3)\bar{u}[\rho u] + [p] + \frac{\gamma-1}{2}[\rho u^2]$$

ie $\quad \frac{3-\gamma}{2}[\rho u^2] = (3-\gamma)\bar{u}[\rho u] - \frac{3-\gamma}{2}\bar{u}^2[\rho]$.

Remark that it is always possible to write, if $u_L \neq u_R$,

$$\bar{u} = \alpha u_L + (1-\alpha)u_R =: \vec{u}$$

while the Leibniz formula states that

$$[XY] = \vec{X}[Y] + \overleftarrow{Y}[X] \quad \text{with} \quad \begin{aligned}\vec{X} &= \alpha X_L + (1-\alpha)X_R \\ \overleftarrow{Y} &= (1-\alpha)Y_L + \alpha Y_R\end{aligned}$$

Then, the second component writes also

$$\frac{3-\gamma}{2}[\rho u^2] = (3-\gamma)\vec{u}^2[\rho] + (3-\gamma)\vec{u}\overleftarrow{\rho}[u] - \frac{3-\gamma}{2}\vec{u}^2[\rho]$$

$$\frac{3-\gamma}{2}[\rho u^2] = \frac{3-\gamma}{2}\vec{u}^2[\rho] + (3-\gamma)\vec{u}\overleftarrow{\rho}[u]$$

or

$$[\rho u^2] = \vec{u}^2[\rho] + 2\vec{u}\overleftarrow{\rho}[u]$$

But $\quad [\rho u^2] = [\rho u]\vec{u} + [u]\overleftarrow{\rho u}$

$$= [\rho]\vec{u}^2 + [u]\overleftarrow{\rho}\vec{u} + [u]\overleftarrow{\rho u}$$

so that we finally get

$$[u](\overleftarrow{\rho}\vec{u} + \overleftarrow{\rho u}) = 2\vec{u}\overleftarrow{\rho}[u]$$

or

$$\overleftarrow{\rho u} = \vec{u}\overleftarrow{\rho}$$

This equation will allow us to define α. Indeed, developing leads to the second order equation

$$(\rho_R - \rho_L)\alpha^2 + 2\alpha\rho_L - \rho_L = 0$$

which admits two roots if $\rho_R \neq \rho_L$:

$$\alpha = \frac{\sqrt{\rho_L}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad \text{and} \quad \alpha = \frac{\sqrt{\rho_L}}{\sqrt{\rho_L} - \sqrt{\rho_R}}$$

and only one root if $\rho_L = \rho_R =: \rho$:

$$\alpha = 1/2 \ .$$

Then, the only continuous way to define α is to set

$$\alpha = \frac{\sqrt{\rho_L}}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

(the other choice gives $\alpha \to \infty$ if $\rho_L \to \rho_R$)

which gives

$$\bar{u} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ .$$

Similar calculations on the third component lead to the definition

$$\bar{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

These are the two only possible choices.

But of course, there exist other Roe-type linearizations based on other strategies, but they do not consist in the Jacobian matrix evaluated on some average state.