Definition

The third equation in (32) is called the Hugoniot equation. It is of purely thermodynamic nature since it involves the thermodynamic variables τ and p only. In other words, it corresponds to the projection onto the (τ, p)-plane of the shock solutions.
The function $H = H(\tau, p)$ with center (τ_0, p_0) defined by

$$H(\tau, p) = \varepsilon(\tau, p) - \varepsilon(\tau_0, p_0) + \frac{1}{2}(p + p_0)(\tau - \tau_0)$$

is called the Hugoniot function, and the graph of $H(\tau, p) = 0$ the Hugoniot curve.
This is the set of all states (τ, p) that can be connected to the state (τ_0, p_0) by a discontinuity satisfying the Rankine-Hugoniot jump conditions (that is not yet necessarily admissible in the entropy sense)

From now on, we focus on the case of a polytropic ideal gas for the sake of simplicity. Recall that in this case

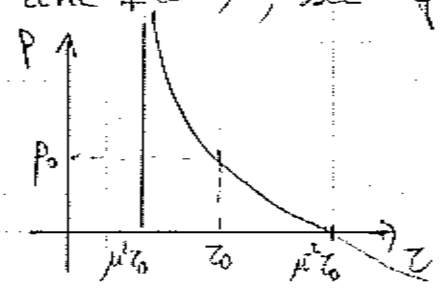
$$(\gamma - 1)\varepsilon = p\tau, \quad c^2 = \gamma p \tau, \quad s = C_v \ln(p\tau^{\gamma}), \quad \gamma > 1,$$

up to an additive constant for s.
Let us first observe that in this case, the Hugoniot curve is given by

$$p = p(\tau) = \frac{\tau_0 - \mu^2 \tau}{\tau - \mu^2 \tau_0} p_0, \quad \mu^2 = \frac{\gamma - 1}{\gamma + 1} > 0, \quad (33)$$

$$\mu^2 < 1$$

after easy calculations.
Note that for $\tau > \mu^{-2}\tau_0$ and $\tau < \mu^2 \tau_0$ the pressure becomes negative.
The part of the Hugoniot curve with physical meaning thus corresponds to $\tau_{min} = \mu^2 \tau_0 < \tau < \tau_{max} = \mu^{-2}\tau_0$ (the pressure p varies between 0 and $+\infty$), see figure below:



In order to eventually consider only admissible discontinuities in the entropy sense, we now exhibit a first property of the Hugoniot curve, stating that s is a decreasing function of τ along the Hugoniot curve.

Lemma

The function $\tau \to s(\tau, p(\tau))$ with $s(\tau, p) = C_v \ln(p\tau^{\gamma})$ and $p(\tau) = \dfrac{\tau_0 - \mu^2\tau}{\tau - \mu^2\tau_0}\, p_0$, $\mu^2 = \dfrac{\gamma - 1}{\gamma + 1}$ is strictly decreasing.

Proof

Easy calculations not reported here lead to

$$\frac{d}{d\tau} s(\tau, p(\tau)) = \frac{\partial s}{\partial \tau}(\tau, p(\tau)) + p'(\tau)\frac{\partial s}{\partial p}(\tau, p(\tau))$$

$$= -\frac{p_0 \gamma \mu^2 (\tau - \tau_0)^2}{p(\tau)\,\tau\,(\tau - \mu^2\tau_0)^2},$$

which gives the expected result.

As a consequence, we have the following entropy selection principle:

Corollary

The entropy inequality, if $\mathbb{V}_0$ (resp. $\mathbb{V}$) is the left (resp. right) state, ie

$$-\sigma(\rho S - \rho_0 S_0) + (\rho S u - \rho_0 S_0 u_0) < 0 \quad \text{with } S = -s$$

$$\iff M(S - S_0) < 0 \qquad (34)$$

$$\iff M(s - s_0) > 0$$

is equivalent to

$$\begin{cases} \text{For } M > 0: & \tau < \tau_0 \iff p > p_0 \iff u < u_0 \\ \text{For } M < 0: & \tau > \tau_0 \iff p < p_0 \iff u > u_0 \end{cases} \qquad (35)$$

Proof

The proof of this result easily follows from (32) and the previous lemma. ∎

The following statement is an equivalence result with Lax inequalities.

Proposition

The entropy selection principle (34) is equivalent to

$$\begin{cases} \text{For } M>0 : & \begin{vmatrix} u-c < \sigma < u \\ \sigma < u_0 - c_0 \end{vmatrix} \\ \text{For } M<0 : & \begin{vmatrix} u+c < \sigma \\ u_0 < \sigma < u_0 + c_0 \end{vmatrix} \end{cases} \qquad (36)$$

Note that this result emphasizes the fact that $M>0$ (respectively $M<0$) is associated with a 1-shock (resp. 3-shock).

Proof

We focus for instance on the case $M = \rho(u-\sigma) > 0$. One thus has to prove that (34) is equivalent to $u-c<\sigma<u_0-c_0$. Let us observe that

$$u-c<\sigma<u_0-c_0$$

$$\Leftrightarrow \begin{cases} (u-\sigma)^2 < \gamma p \tau \\ (u_0-\sigma)^2 > \gamma p_0 \tau_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} M^2\tau^2 < \gamma p \tau \\ M^2\tau_0^2 > \gamma p_0 \tau_0 \end{cases} \qquad \text{since } \begin{cases} M\tau = (u-\sigma) \\ M\tau_0 = (u_0-\sigma) \end{cases}$$

$$\Leftrightarrow \gamma p_0/\tau_0 < M^2 < \gamma p/\tau \qquad \text{with } M^2 = -\frac{p-p_0}{\tau-\tau_0}$$

Let us now consider the functions $\tau \to M^2(\tau) = -\frac{p(\tau)-p_0}{\tau-\tau_0}$ and $\tau \to N(\tau) = \frac{\tau M^2(\tau)}{\gamma p(\tau)}$ with $p(\tau) = \frac{\tau_0-\mu^2\tau}{\tau-\mu^2\tau_0} p_0$. Actually, these functions M and N can be parametrized by the ratio $\eta = \tau/\tau_0$. More precisely and with a little abuse in the notations, we have

$$M^2(\tau) \equiv M^2(\eta) = \frac{(1+\mu^2)p_0}{\tau_0(\eta-\mu^2)} \quad \text{and} \quad N(\tau) \equiv N(\eta) = \frac{\eta(1+\mu^2)}{\gamma(1-\mu^2\eta)}.$$

Easy calculations lead to

$$\frac{d}{d\eta} \Pi^2(\eta) = -\frac{(1+\mu^2)p_0}{\tau_0(\eta-\mu^2)^2} < 0$$

$$\frac{d}{d\eta} N(\eta) = \frac{(1+\mu^2)}{\gamma(1-\mu^2\eta)^2} > 0$$

which means that the functions $\Pi^2(\eta)$ and $N(\eta)$ are respectively decreasing and increasing with respect to η.

Recall that our objective is to prove that

$$\gamma p_0/\tau_0 < \Pi^2 < \gamma p/\tau \quad \text{with } \Pi^2 = -\frac{p-p_0}{\tau-\tau_0}$$

that is
$$\begin{cases} \Pi^2(\eta) > \Pi^2(\eta=1) \\ N(\eta) < N(1) \end{cases} \quad \text{since } \gamma = \frac{1+\mu^2}{1-\mu^2},$$

is equivalent to $\tau < \tau_0$. This is clearly true since $\eta = \tau/\tau_0$ and by the monotonicity properties on Π^2 and N. ∎

Remark.

Let us define the Mach number $\mathcal{M} = \frac{|u-\sigma|}{c}$. The last proposition says that for:

- an admissible 1-shock we have

$$\mathcal{M}(W) < 1 < \mathcal{M}(W_0)$$

- an admissible 3-shock we have

$$\mathcal{M}(W_0) < 1 < \mathcal{M}(W)$$

Which means that for a 1-shock (respectively 3-shock), the left (resp. right) state is supersonic while the right (resp. left) state is subsonic, or equivalently that the upstream state is supersonic and the downstream state is subsonic.

Collecting all the results obtained in this paragraph, we are now able to give the projections in the (p,u)-plane of the 1- and 3-shock curves. Observing that (33) writes also with clear notations

$$\tau = \tau(p) = \tau_0 + \frac{\tau_0 (p-p_0)(\mu^2-1)}{p+\mu^2 p_0},$$

we have

$$\mathcal{S}_1(\mathbb{V}_0) = \left\{ \mathbb{V} \in \Omega \,/\, p > p_0\,,\ \tau = \tau(p),\ u = u_0 + M(p)(\tau(p)-\tau_0) \right\}$$

$$\left(= \left\{ \mathbb{V} \in \Omega \,/\ \mathbb{V}_0 \xrightarrow{\text{admissible 1-shock}} \mathbb{V} \right\} \right)$$

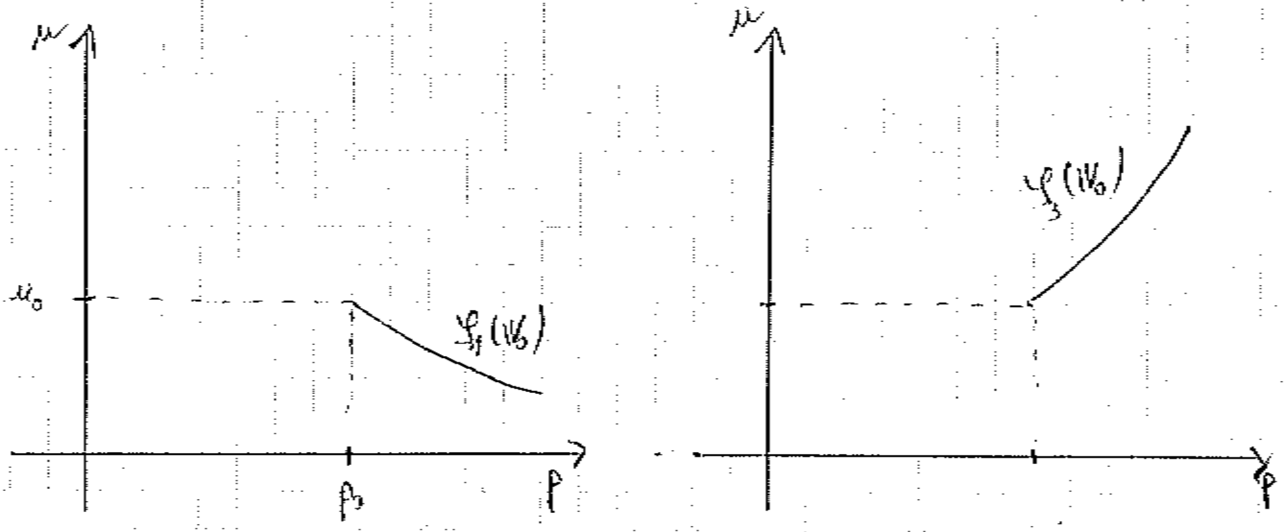
with $M(p) = \sqrt{-\dfrac{p-p_0}{\tau(p)-\tau_0}} > 0$ and

$$\mathcal{S}_3(\mathbb{V}_0) = \left\{ \mathbb{V} \in \Omega \,/\, p > p_0\,,\ \tau(p),\ u = u_0 + M(p)(\tau(p)-\tau_0) \right\}$$

$$\left(= \left\{ \mathbb{V} \in \Omega \,/\ \mathbb{V} \xleftarrow{\text{admissible 3-shock}} \mathbb{V}_0 \right\} \right)$$

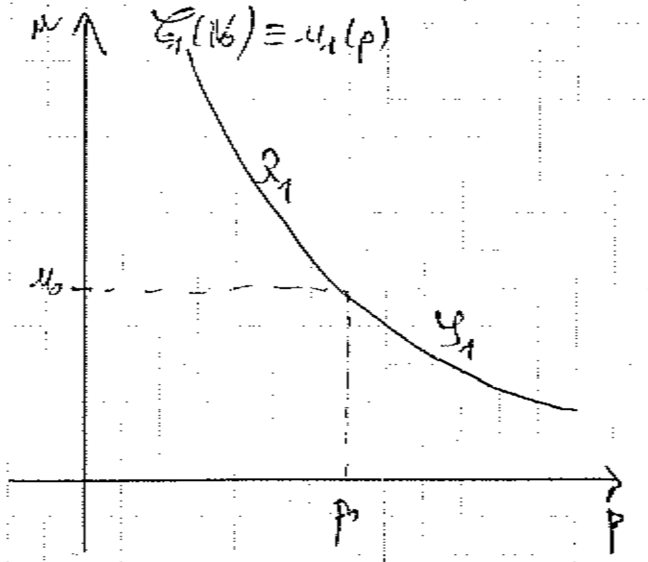
with $M(p) = -\sqrt{-\dfrac{p-p_0}{\tau(p)-\tau_0}} < 0$.

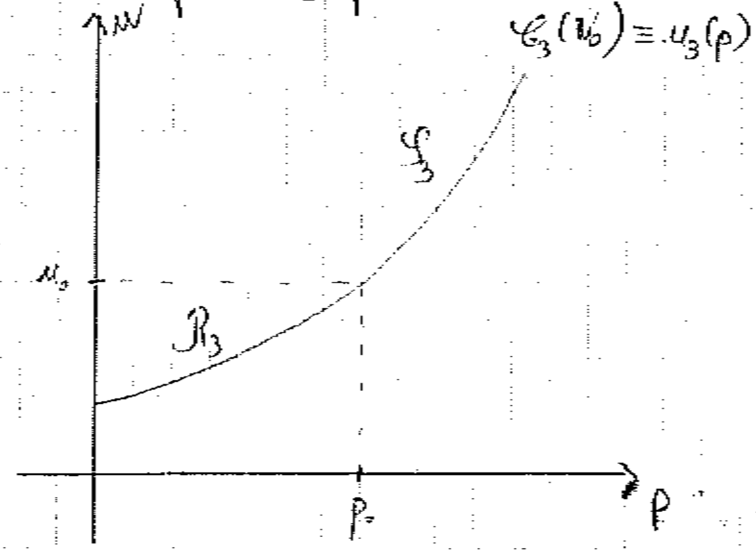
One can actually prove that the function $u = u(p)$ is decreasing for a 1-shock and increasing for a 3-shock, so that in the (p,u) plane, $\mathcal{S}_1(\mathbb{V}_0)$ and $\mathcal{S}_3(\mathbb{V}_0)$ look like:



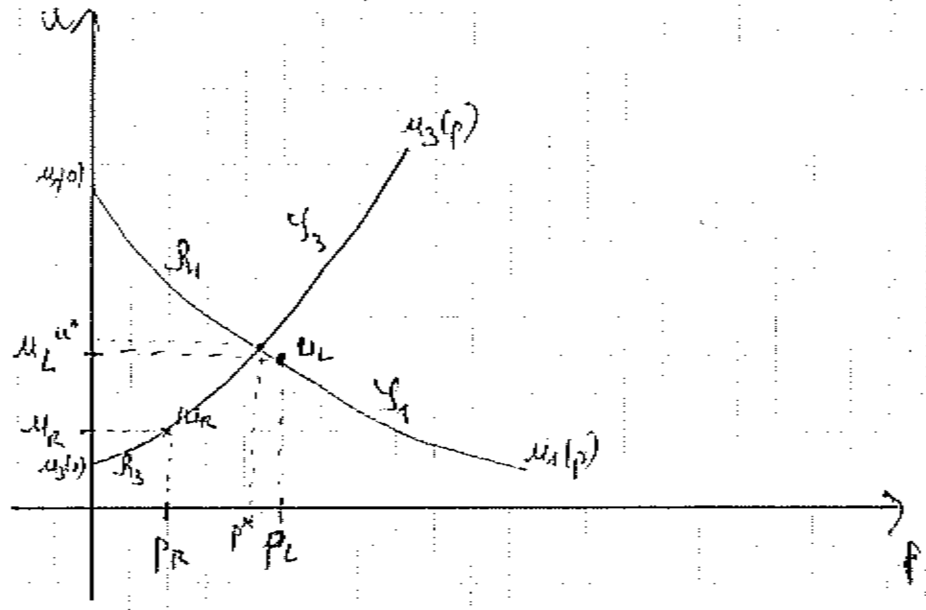
We are now tempted to define $\mathcal{C}_1(\mathbb{V}_0) = \mathcal{R}_1(\mathbb{V}_0) \cup \mathcal{S}_1(\mathbb{V}_0)$ and $\mathcal{C}_3(\mathbb{V}_0) = \mathcal{R}_3(\mathbb{V}_0) \cup \mathcal{S}_3(\mathbb{V}_0)$. These curves give in the (p,u)-plane all the states that can be join on the right (resp. left) to $\mathbb{V}_0$

by a 1-wave (resp. 3-wave) which can be either a rarefaction wave or an admissible shock. We have the following forms:





In order to solve the Riemann problem (8)-(24) associated with the left and righ states U_L and U_R, it is now a matter to find the intersection of $\mathcal{C}_1(U_L)$ and $\mathcal{C}_3(U_R)$ (still with a little abuse in the notations):



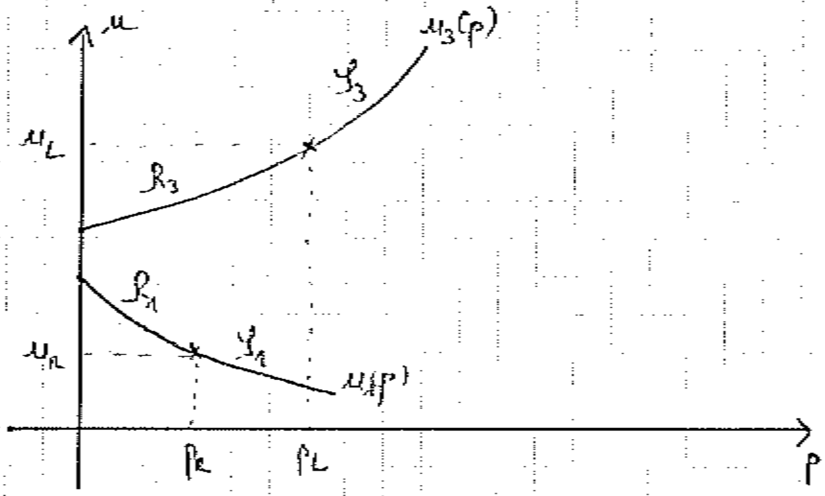
Geometrically, it is then clear that a necessary and sufficient condition for (u^*, ρ^*) to be uniquely defined is

$$u_3(0) < u_1(0).$$

In the case of the above figure, the pattern of the Riemann solution is as follows:

$$\mathbb{U}_L \xrightarrow{\text{1-rarefaction}} \mathbb{U}_L^* \xrightarrow{\substack{\text{contact}\\ \text{discontinuity}}} \mathbb{U}_R^* \xrightarrow{\text{3-shock}} \mathbb{U}_R$$

In the case $u_3(0) > u_1(0)$



vacuum appears and we have the following pattern.

$$\mathbb{U}_L \xrightarrow{\text{1-rarefaction}} \begin{matrix}\rho=0\\ \rho=0\end{matrix} \xrightarrow{\text{3-rarefaction}} \mathbb{U}_R$$