Regarding the second characteristic field, (22) gives

$$\frac{\partial p}{\partial s}\frac{\partial w}{\partial \rho} - c^2 \frac{\partial w}{\partial s} = 0$$

or equivalently $$\frac{\partial p}{\partial s}\frac{\partial w}{\partial \rho} - \frac{\partial p}{\partial \rho}\frac{\partial w}{\partial s} = 0$$

Then u and p are clearly candidates. There gradients are given by

$$\nabla u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nabla p = \begin{pmatrix} \partial p/\partial \rho \\ 0 \\ \partial p/\partial s \end{pmatrix}$$

and are clearly linearly independent.

To sum up, we have checked that the following three pairs are Riemann invariants for the three characteristic fields of the Euler equations:

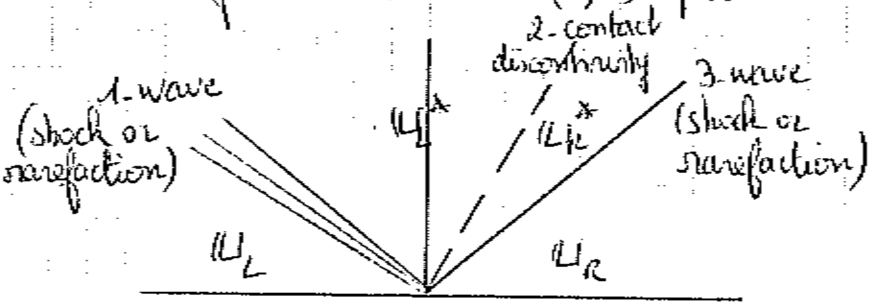
$$(u+\ell, s), \quad (u, p), \quad (u-\ell, s) \qquad (23)$$

Solution of the Riemann problem

Recall first that the model under consideration is strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. The Lax theorem then ensures that there exists a unique solution to (8) for the Riemann initial data

$$\mathbb{U}(x,0) = \begin{cases} \mathbb{U}_L & \text{if } x < 0 \\ \mathbb{U}_R & \text{if } x > 0, \end{cases} \qquad (24)$$

provided that $\mathbb{U}_L$ and $\mathbb{U}_R$ are sufficiently close. Here, we have used the notation $\mathbb{U} = (\rho, \rho u, \rho E)$. The form of the Riemann solution is as follows in the (x,t)-plane



$\lambda_1 = u - c$ (VNL)

$\lambda_2 = u$ (LD)

$\lambda_3 = u + c$ (VNL)

Our objective is to extend this result without the closeness assumption, and to show how the intermediates states $\mathbb{U}_L^*$ and $\mathbb{U}_R^*$ can be determined in practice.
In addition to (19), we will assume that
$$\frac{\partial \tilde{p}}{\partial S}(\tau, S) > 0 \qquad (25)$$
so that the changes of variables $\mathbb{V} = (\rho, u, p)$ and $\mathbb{W} = (S, u, p)$ are admissible. Note however that we will be soon focusing on the particular case of a polytropic ideal gas for the sake of simplicity.

<u>Contact discontinuity curve</u>

The Riemann invariants associated with the second characteristic field are u and p. Then, with clear notations we have
$$\mathcal{X}_2(\mathbb{V}_0) = \{ (\rho, u, p) \,/\, \rho > 0,\ u = u_0,\ p = p_0 \} \qquad (26)$$
$\mathcal{X}_2(\mathbb{V}_0, \varepsilon)$ $\quad \rho_0 + \varepsilon$

In the (p,u)-plane, contact discontinuities are then represented by a single point. In other words, the states $\mathbb{U}_L^*$ and $\mathbb{U}_R^*$ we are looking for will be represented by the same point in the (p, u)-plane. Since $\mathbb{U}_L^*$ (resp $\mathbb{U}_R^*$) is connected on the right (resp. left) to $\mathbb{U}_L$ (resp $\mathbb{U}_R$), we will try in the next calculations to project in the (p,u)-plan the 1-wave curve (resp. 3-wave curve) made of all the states that can be joined to $\mathbb{U}_L$ (resp $\mathbb{U}_R$) on the right (resp. left) by an admissible shock or by a rarefaction fan associated with the first (resp third) eigenvalue, and to intersect these 1- and 3-wave curves in order to get
$$u^* := u_L^* = u_R^*, \quad p^* := p_L^* = p_R^*$$

so that

$$\begin{cases} \nabla\lambda_1 \cdot r_1 = -\rho \dfrac{\partial c}{\partial \rho} - c = -\left(c+\rho \dfrac{\partial c}{\partial \rho}\right), \\ \nabla\lambda_2 \cdot r_2 = 0, \\ \nabla\lambda_3 \cdot r_3 = c+\rho \dfrac{\partial c}{\partial \rho} \end{cases} \quad (21)$$

Using (19)', we then obtain that the first and the third characteristic fields are genuinely nonlinear while the second one is linearly degenerate.

Riemann Invariants

For each characteristic field, we seek $(p-1) = 2$ Riemann invariant whose gradients are linearly independent. Recall that a k-Riemann invariant $w : \Omega \to \mathbb{R}$ is defined by

$$\nabla w(u) \cdot r_k(u) = 0 \quad \forall u. \qquad (22)$$

We choose here again to work with the set of variables (ρ, u, s).
Let us begin with the first characteristic field. (22) gives

$$\rho \frac{\partial w}{\partial \rho} - c \frac{\partial w}{\partial u} = 0$$

Then, s and $u + l(\rho, s)$ with l defined up to an additive function of s by

$$\frac{\partial l}{\partial \rho}(\rho, s) = \frac{c(\rho, s)}{\rho}$$

are clearly candidates. It remains to check that these gradients are linearly independent. We have

$$\nabla s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \nabla(u+l) = \begin{pmatrix} c/\rho \\ 1 \\ \dfrac{\partial l}{\partial s} \end{pmatrix}$$

These vectors are clearly independent.
In a similar way, we get that s and $u - l(\rho, s)$ are two Riemann invariants with linearly independent gradients associated with the third characteristic field.

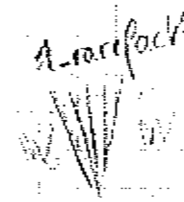
Rarefaction curves

The Riemann invariants associated with the first and third characteristic fields are respectively given by $(u+\ell, s)$ and $(u-\ell, s)$ with ℓ defined up to an additive constant by

$$\ell(\rho, s) = \int^{\rho} \frac{c(r,s)}{r}\, dr$$

We thus have

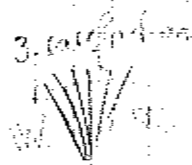
$$R_1(W_0) = \left\{ W \in \Omega \,/\, W_0 \xrightarrow{1\text{-rarefaction}} W \right\}$$



$$= \left\{ W \in \Omega \,/\, s = s_0,\ u = u_0 - \int_{\rho_0}^{\rho(p,s_0)} \frac{c(r,s_0)}{r}\, dr, \quad \lambda_1(W) = u - c > \lambda_1(W_0) = u_0 - c_0 \right\} \tag{27}$$

and

$$R_3(W_0) = \left\{ W \in \Omega \,/\, W \xrightarrow{3\text{-rarefaction}} W_0 \right\}$$



$$= \left\{ W \in \Omega \,/\, s = s_0,\ u = u_0 + \int_{\rho_0}^{\rho(p,s_0)} \frac{c(r,s_0)}{r}\, dr, \quad \lambda_3(W) = u + c < \lambda_3(W_0) = u_0 + c_0 \right\}. \tag{28}$$

Let us focus on $R_1(W_0)$ ($R_3(W_0)$ being treated similarly).
We have

$$u = u(p) = u_0 - \int_{\rho_0}^{\rho(p,s_0)} \frac{c(r,s_0)}{r}\, dr$$

so that $u'(p) = -\frac{\partial \rho}{\partial p} \times \frac{c(\rho(p,s_0),s_0)}{\rho(p\ s_0)}$

But by assumption $\frac{\partial p}{\partial \tau}(\tau, s) < 0$ so that $\frac{\partial p}{\partial \rho}(\rho, s) > 0$ (see (19)') and $u'(p) < 0$ follows. In the (p,u)-plane, the 1-rarefaction curve is then strictly decreasing.

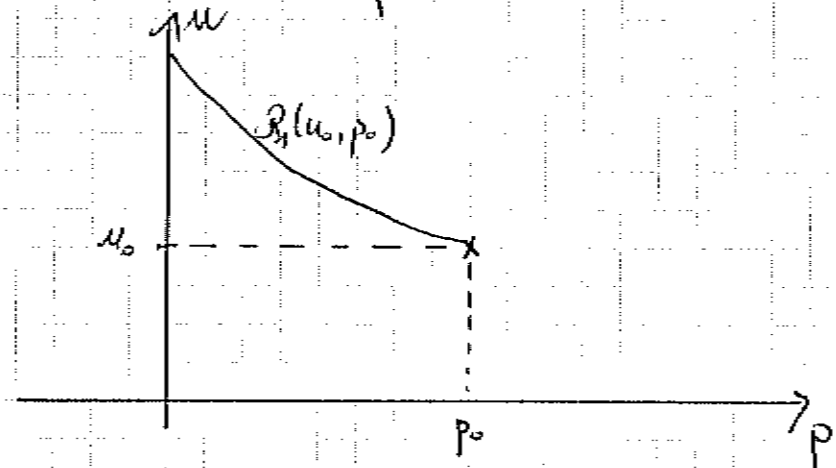
On the other hand, we have

$$\frac{d}{dp}\left\{ u(p) - c(\rho(p,s_0),s_0) \right\} = \frac{d}{dp}\left\{ u(p) - \sqrt{\frac{\partial p}{\partial \rho}(\rho(p,s_0),s_0)} \right\}$$

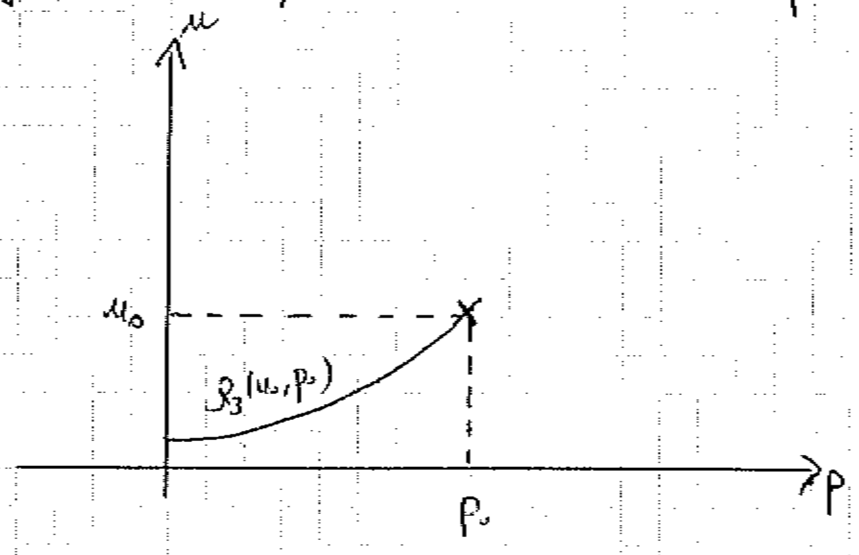
$$= u'(p) - \frac{\frac{\partial \rho}{\partial p} \times \frac{\partial^2 p}{\partial \rho^2}}{2c} = -\frac{\partial \rho}{\partial p} \times \frac{c}{\rho} - \frac{\partial \rho}{\partial p} \times \frac{\partial^2 p/\partial \rho^2}{2c} < 0$$

since $\frac{\partial^2 p}{\partial \tau^2}(\tau,s) > 0$ by assumption (19).

Then, $u - c \geq u_0 - c_0$ if and only if $p \leq p_0$. In the (p,u)-plane the 1-rarefaction curve thus looks like:



Similarly, one can prove that the 3-rarefaction curve looks like



Shock curves

The Rankine-Hugoniot jump relations, for a discontinuity separating two constant states W_0 and W and propagating at velocity σ, writes here as follows:

$$\begin{cases} -\sigma(\rho-\rho_0) + (\rho u - \rho_0 u_0) = 0 \\ -\sigma(\rho u - \rho_0 u_0) + ((\rho u^2 + p) - (\rho_0 u_0^2 + p_0)) = 0 \\ -\sigma(\rho E - \rho_0 E_0) + ((\rho E u + p u) - (\rho_0 E_0 u_0 + p_0 u_0)) = 0 \end{cases} \qquad (29)$$

It will be convenient to introduce the flow velocity $v = u - \sigma$ relative to the discontinuity, and to recast (29) as follows:

$$\begin{cases} \rho_0 v_0 = \rho v \\ \rho_0 v_0^2 + p_0 = \rho v^2 + p \\ \rho_0 E_0 v_0 + p_0 v_0 = \rho E v + p v \end{cases} \qquad (30)$$

The proof of (30) is left to the reader.

Let us now introduce the mass flux M through the discontinuity. We then have

$$\begin{cases} M = \rho v = \rho_0 v_0 \\ M(v - v_0) + (p - p_0) = 0 \\ M(E - E_0) + (pu - p_0 u_0) = 0 \end{cases} \qquad (31)$$

In the following, we will have to distinguish between the three cases $M = 0$, $M > 0$ and $M < 0$. Indeed, let us first observe that $M = 0$ is equivalent to $u = u_0 = \sigma$ and $p = p_0$ which correspond to the case of a contact discontinuity already treated above.

Let us then assume $M \neq 0$ in order to deal with a shock discontinuity, that may be a 1-shock or a 3-shock.

In the neighborhood of V_0, we have with clear notations

$$\begin{cases} V = V_0 + \varepsilon\, r_k(V_0) + O(\varepsilon^2), \\ \sigma = \lambda_k(V_0) + \dfrac{\varepsilon}{2} + O(\varepsilon^2), \end{cases} \qquad (\nabla\lambda_k \cdot r_k = 1)$$

so that

$$M = \rho_0(u_0 - \sigma) = \rho_0(u_0 - \lambda_k(V_0) + O(\varepsilon))$$
$$= \begin{cases} \rho_0 c_0 + O(\varepsilon) & \text{if } k = 1 \\ -\rho_0 c_0 + O(\varepsilon) & \text{if } k = 3. \end{cases}$$

It is then clear that the case $M>0$ (respectively $M<0$) corresponds to a 1-shock (resp. a 3-shock). In other words, for a 1-shock (resp a 3-shock) the flow crosses the shock from the left to the right (resp from the right to the left).

<u>Lemma</u>

For $M \neq 0$, the Rankine-Hugoniot jump relations (30) (or (31)) can be equivalently written

$$\begin{cases} M = \dfrac{u-u_0}{\tau-\tau_0} \\ M^2 = -\dfrac{p-p_0}{\tau-\tau_0} \\ \varepsilon - \varepsilon_0 + \dfrac{1}{2}(p+p_0)(\tau-\tau_0) = 0 \end{cases} \quad (32) \qquad p = p(\tau, \varepsilon)$$

<u>Proof</u>

By definition of M, we first clearly have $M\tau = v = u - \sigma$ and $M\tau_0 = v_0 = u_0 - \sigma$ so that $M(\tau-\tau_0) = u - u_0$. Note that we implictly assume that $p_0 \neq p$ since otherwise we easily get $u = u_0$, $p = p_0$ and then $V = V_0$. This case is not interesting since it means that the discontinuity does not excist. The first relation of (32) is thus proved. The second relation of (32) then follows since by (31), we know that $M(v - v_0) = -(p-p_0)$ or equivalently $M(u-u_0) = -(p-p_0)$, so that $M^2(\tau-\tau_0) = -(p-p_0)$.

Let us now focus on the third one. (31) gives

$$M\left(\left(\varepsilon + \frac{u^2}{2}\right) - \left(\varepsilon_0 + \frac{u_0^2}{2}\right)\right) + (pu - p_0u_0) = 0$$

or $\quad M(\varepsilon - \varepsilon_0) + \dfrac{M}{2}(u-u_0)(u+u_0) + (pu - p_0u_0) = 0$

But $\quad pu - p_0u_0 = \dfrac{1}{2}(p+p_0)(u-u_0) + \dfrac{1}{2}(u+u_0)(p-p_0)$

$$= \frac{M}{2}(p+p_0)(\tau-\tau_0) - \frac{M^2}{2}(u+u_0)(\tau-\tau_0)$$

$$= \frac{M}{2}(p+p_0)(\tau-\tau_0) - \frac{M}{2}(u+u_0)(u-u_0)$$

(Conversely, if we set $\sigma = u - M\tau$, it is easy to see that $(32) \Rightarrow (30)$)

so that we get $M(\varepsilon - \varepsilon_0) + \dfrac{M}{2}(p+p_0)(\tau-\tau_0) = 0$.

This concludes the proof since $M \neq 0$ by assumption. ∎