The gas dynamics equations in Eulerian coordinates

A system of partial differential equations of specific interest in the applications is the gas dynamics equations which govern the evolution of a compressible fluid. In Eulerian coordinates, they write

$$\begin{cases} \partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho u_j) = 0 \\ \partial_t \rho u_i + \sum_{j=1}^{3} \partial_{x_j}(\rho u_i u_j + p\delta_{ij}) = 0 \quad i = 1,2,3 \\ \partial_t \rho E + \sum_{j=1}^{3} \partial_{x_j}(\rho E u_j + p u_j) = 0 \end{cases} \qquad (1)$$

Here we note that $d=3$, $p=5$, while $\rho > 0$ denotes the density of the fluid, $u = (u_1, u_2, u_3)^t$ the velocity, p the pressure, $E = \frac{|u|^2}{2} + \varepsilon$ the specific total energy, ε the specific internal energy. In order to close this system, an incomplete equation of state of the form

$$p = p(\rho, \varepsilon) \qquad (2)$$

can be prescribed. Here, we supplement (1) with a complete equation of state, which means that we assume in addition the existence of an integrand factor

$$T = T(\rho, \varepsilon) > 0 \qquad (3)$$

called the temperature of the fluid such that

$$\mathcal{X}(\rho, \varepsilon) = \mathcal{X}(\tau, \varepsilon)$$

$$ds = \frac{1}{T}(d\varepsilon + p\, d\tau), \quad \tau = 1/\rho, \qquad (4)$$

ie by the Poincaré's theorem,

$$\frac{\partial(1/T)}{\partial \tau} = \frac{\partial(p/T)}{\partial \varepsilon}. \qquad (5)$$

The mapping $(\tau, \varepsilon) \to s(\tau, \varepsilon)$ (τ is called the covolume) is called the specific thermodynamic entropy*. We assume in addition that this function is strictly concave in (τ, ε).

<u>Example</u>

* : the so-called physical entropy

We will typically focus on the case of a polytropic ideal gas (thermal-

-ly and calorically perfect), we have

$$p = (\gamma - 1)\rho\varepsilon \quad \text{and} \quad \varepsilon = C_v T \qquad (6)$$

where γ is the constant adiabatic coefficient ($\gamma > 1$) and C_v the specific heat at constant volume. Easy calculations give

$$s = C_v \ln\left(\frac{\varepsilon}{\rho^{\gamma-1}}\right) + s_0 \quad . \qquad (7)$$

In the following, we will focus without restriction on the 1D case ($d = 1$)

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x (\rho E u + p u) = 0 \end{cases} \qquad (8)$$

with $E = \varepsilon + \frac{u^2}{2}$ and still (2)-(3)-(4).

<u>Relation between the thermodynamic entropy and the mathematical entropy</u>.

We start with the following result.

<u>Proposition</u>

Smooth solutions of (8) obey the following additional conservation law

$$\partial_t \rho s + \partial_x \rho s u = 0 . \qquad (9)$$

<u>Proof</u>

Our first objective is to prove that

$$\partial_t \tau + u \partial_x \tau - \tau \partial_x u = 0$$

$$\text{and} \quad \partial_t \varepsilon + u \partial_x \varepsilon + p\tau \partial_x u = 0 . \qquad (10)$$

The mass conservation equation writes

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

which gives, after multiplication by $-1/\rho^2$:

$$\partial_t \tau + u \partial_x \tau - \tau \partial_x u = 0 .$$

Next, the energy conservation equation writes

$$\rho\partial_t E + E\partial_t\rho + E\partial_x\rho u + \rho u\partial_x E + p\partial_x u + u\partial_x p = 0$$

so that, since $\partial_t\rho + \partial_x\rho u = 0$:

$$\partial_t E + u\partial_x E + p\tau\partial_x u + \tau u\partial_x p = 0. \quad (11)$$

Next, the momentum conservation equation gives following the same idea

$$\partial_t u + u\partial_x u + \tau\partial_x p = 0.$$

Multiplying by u gives

$$\partial_t \frac{u^2}{2} + u\partial_x \frac{u^2}{2} + \tau u\,\partial_x p = 0. \quad (12)$$

(11)-(12) give

$$\partial_t \mathcal{E} + u\partial_x\mathcal{E} + p\tau\partial_x u = 0$$

so that (10) is now proved.

But (10) easily gives

$$\frac{1}{T}\left(\partial_t\mathcal{E} + p\partial_t\tau\right) + \frac{u}{T}\left(\partial_x\mathcal{E} + p\partial_x\tau\right) = 0$$

that is, since $ds = \frac{1}{T}(d\mathcal{E} + p\,d\tau)$:

$$\partial_t s + u\partial_x s = 0.$$

or again $\rho\partial_t s + \rho u\partial_x s = 0$. The mass conservation equation $\partial_t\rho + \partial_x\rho u = 0$ then gives the expected result ∎

In addition, one can prove (see for instance the book by E. Godlewski and P.A. Raviart)

Proposition

The mapping $(\rho, \rho u, \rho E) \to -\rho s\left(\frac{1}{\rho}, \frac{\rho E}{\rho} - \frac{1}{2}\frac{(\rho u)^2}{\rho}\right)$ is strictly convex.

As a conclusion, Propositions ? and ? allow to state that the function $S = -\rho s$ is a (mathematical) entropy to (1) in the sense of Definition ?, with entropy flux $G = Su$

Since S is strictly convex, one deduces that system (8) is

symmetrizable and then hyperbolic.
Hyperbolicity is proved but we have at this stage no idea of the corresponding eigenvalues. This is the matter of the next paragraph.

Eigenstructure

In order to calculate the eigenstructure of (8), we propose to consider the admissible change of variable $(\rho, \rho u, \rho E) \to (\rho, u, s)$, which leads to

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho \partial_x u = 0 \\ \partial_t u + u\partial_x u + \tau \partial_x p = 0 \\ \partial_t s + u\partial_x s = 0, \end{cases} \quad (13)$$

or equivalently, setting $p = \bar{p}(\rho, s)$ with a little abuse in the notations,

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho \partial_x u = 0 \\ \partial_t u + u\partial_x u + \tau \partial_\rho \bar{p}\, \partial_x \rho + \tau \partial_s \bar{p}\, \partial_x s = 0 \\ \partial_t s + u\partial_x s = 0. \end{cases} \quad (14)$$

Then, the "Jacobian" matrix to be considered writes

$$\begin{pmatrix} u & \rho & 0 \\ \tau \partial_\rho \bar{p} & u & \tau \partial_s \bar{p} \\ 0 & 0 & u \end{pmatrix} \quad (15)$$

We easily obtain the real distinct eigenvalues

$$\lambda_1 = u - c < \lambda_2 = u < \lambda_3 = u + c \quad (16)$$

where the "sound speed" c is given by

$$c = \sqrt{\partial_\rho \bar{p}} = \sqrt{-\tau^2 \partial_\tau \tilde{p}} \quad (17)$$

$p = p(\rho, \varepsilon)$
$\tilde{p} = \tilde{p}(\tau, s)$

Note that the strict convexity of the function $(\tau, \varepsilon) \to -s(\tau, \varepsilon)$ (which holds true here by assumption) is equivalent to the strict convexity of the function $(\tau, s) \to \varepsilon(\tau, s)$ so that the relation

$$d\varepsilon = T ds - \tilde{p} d\tau$$

gives in particular that $-\frac{\partial \tilde{p}}{\partial \tau} > 0$ which proves again that the sound speed is actually real.

Exercise

Prove that in the case of a polytropic ideal gas we have

$$c^2 = \gamma p / \rho .$$

The eigenvectors associated with λ_1, λ_2 and λ_3 are easily found to be given by

$$r_1 = \begin{pmatrix} \rho \\ -c \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} \partial \tilde{p}/\partial s \\ 0 \\ -c^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} \rho \\ c \\ 0 \end{pmatrix} \qquad (18)$$

Nature of the characteristic fields.

We assume that the pressure $\tilde{p} = \tilde{p}(\tau, s)$ is a strictly decreasing and strictly convex function in τ:

$$\frac{\partial \tilde{p}}{\partial \tau}(\tau, s) < 0, \quad \frac{\partial^2 \tilde{p}}{\partial \tau^2}(\tau, s) > 0 \qquad (19)$$

or equivalently

$$\frac{\partial p}{\partial \rho}(\rho, s) = c^2 > 0, \quad \frac{\partial^2 p}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial p}{\partial \rho} = \frac{2c}{\rho}\left(\rho \frac{\partial c}{\partial \rho} + c\right) > 0. \qquad (19)'$$

These assumptions are realistic and we note that the first inequality is nothing but a consequence of the strict convexity of $\varepsilon(\tau, s)$ ensuring that the sound speed is real.

Next, differentiating with respect to $(\rho, u, s)^t$ leads to.

$$\nabla \lambda_1 = \begin{pmatrix} -\frac{\partial c}{\partial \rho} \\ 1 \\ -\frac{\partial c}{\partial s} \end{pmatrix}, \quad \nabla \lambda_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nabla \lambda_3 = \begin{pmatrix} \frac{\partial c}{\partial \rho} \\ 1 \\ \frac{\partial c}{\partial s} \end{pmatrix} \qquad (20)$$