Contact discontinuities

We can state the following result.

Proposition

If the k-characteristic field is linearly degenerate, the curve $\mathcal{S}_k(u_0)$ is an integral curve of the vector field r_k, and we have

$$\sigma(u_0, \Psi_k(\varepsilon)) = \lambda_k(\Psi_k(\varepsilon)) = \lambda_k(u_0). \qquad (31)$$

Moreover, we have for any k Riemann invariant w:

$$w(\Psi_k(\varepsilon)) = w(u_0). \qquad (32)$$

Proof

Let us consider the integral curve of the vector field r_k passing through the point u_0, ie the solution $\xi \to v(\xi)$ of

$$\begin{cases} v'(\xi) = r_k(v(\xi)) \\ v(0) = u_0 \end{cases}$$

Let us first note that by linear degeneracy

$$\frac{d}{d\xi}\lambda_k(v(\xi)) = \nabla\lambda_k(v(\xi)) \cdot v'(\xi) = \nabla\lambda_k(v(\xi)) \cdot r_k(v(\xi)) = 0$$

so that λ_k is constant along the integral curve, ie $\lambda_k(v(\xi)) = \lambda_k(u_0)$.

Next, let us check that the Rankine-Hugoniot condition holds along the integral curve with constant speed $\sigma(u_0, u) = \lambda_k(u_0)$. Again by linear degeneracy we have

$$\frac{d}{d\xi}\left\{ f(v(\xi)) - f(u_0) - \lambda_k(v(\xi))(v(\xi) - u_0) \right\}$$

$$= \left(A(v(\xi)) - \lambda_k(v(\xi))\, Id \right) \underbrace{v'(\xi)}_{= r_k(v(\xi))} - \nabla\lambda_k(v(\xi)) \cdot \underbrace{v'(\xi)}_{= r_k(v(\xi))} (v(\xi) - u_0)$$

$$= 0$$

and therefore $\quad f(v(\xi)) - f(u_0) = \lambda_k(v(\xi))(v(\xi) - u_0)$ or

$$f(v(\xi)) - f(u_0) = \lambda_k(u_0)(v(\xi) - u_0)$$

Since this equality is valid for $\xi = 0$.

Hence the integral curve coincides with $S_k(u_0)$ with

$$\sigma(u_0, v(\xi)) = \lambda_k(v(\xi)) = \lambda_k(u_0).$$

At last, any k-Riemann invariant is known to be constant on an integral curve of r_k, as we have already observed. This concludes the proof. ▫

Thus, to conclude this paragraph and assuming that the kth characteristic field is linearly degenerate, a weak solution of the form (23), ie

$$u(x,t) = \begin{cases} u_0 & \text{if } \frac{x}{t} < \sigma, \\ u_1 & \text{if } \frac{x}{t} > \sigma, \end{cases}$$

with $u_1 \in \mathcal{S}_k(u_0)$, or equivalently $u_0 \in \mathcal{S}_k(u_1)$, and

$$\sigma = \lambda_k(u_0) = \lambda_k(u_1)$$

is called a k-contact discontinuity. Moreover, u_0 and u_1 are such that for any k-Riemann invariant w

$$w(u_0) = w(u_1).$$

<u>Shocks</u>

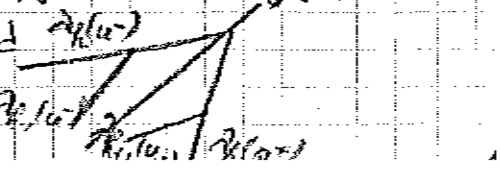
As in the scalar case, it is expected here that not any state of $\mathcal{S}_k(\varepsilon)$ with ε such that $|\varepsilon| \leq \varepsilon_1$ will be admissible (we have used here the same notations as in the lemma ??) when the kth characteristic field is genuinely nonlinear. Let us introduce the following definition.

<u>Definition</u>

A discontinuity between two states u_- (on the left) and u_+ (on the right) and propagating with speed σ such that the Rankine-Hugoniot relations are satisfied satisfies the Lax entropy conditions if there exists $k \in \{1, \dots, p\}$ such that $u_+ \in \mathcal{S}_k(u_-)$ and

$$\lambda_k(u_+) < \sigma < \lambda_{k+1}(u_+) \quad (33)$$

$$\lambda_{k-1}(u_-) < \sigma < \lambda_k(u_-)$$

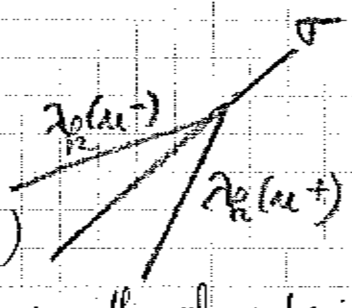


Such a discontinuity is said to be admissible in the sense of Lax.

Then using the parametrization of lemma ?, we define $\mathcal{S}_k^a(u_0)$ as the set of states $\Psi_k(\varepsilon) \in \mathcal{S}_k(u_0)$ that can be connected on the right to u_0 by a k-discontinuity wave that satisfies the Lax entropy conditions.

<u>Remark</u>

Note that (33) implies that



$$\lambda_k(u_+) < \sigma < \lambda_k(u_-) \quad (34)$$

so that in agreement with the scalar theory, the characteristic speeds enter the shock. The shock is said to be compressive.

<u>Proposition</u>

Consider the normalization (16). If the k-th characteristic field is genuinely nonlinear, the curve $\mathcal{S}_k^a(u_0)$ consists of the states $\Psi_k(\varepsilon) \in \mathcal{S}_k(u_0)$ that satisfy

$$\varepsilon \leq 0, \quad |\varepsilon| \leq \varepsilon_1 \text{ small enough.}$$

<u>Proof</u>

Let us set $u(\varepsilon) = \Psi_k(\varepsilon)$ and $\sigma(\varepsilon) = \sigma(u_0, \Psi_k(\varepsilon))$ so that we have invoking the normalization (16):

$$\begin{cases} u(\varepsilon) = u_0 + \varepsilon\, r_k(u_0) + O(\varepsilon^2) \\ \sigma(\varepsilon) = \lambda_k(u_0) + \frac{\varepsilon}{2} + O(\varepsilon^2) \end{cases} \quad (35)$$

We want u_0 and $u(\varepsilon)$ be such that (33) hold true. It is first clear by the second equation of (35) that in order to get the inequality $\sigma(\varepsilon) < \lambda_k(u_0)$, ε sufficiently small must be negative.

Let us now check the other inequalities. Compare $\sigma(\varepsilon)$ and $\lambda_k(u(\varepsilon))$.

We have
$$\begin{aligned} \lambda_k(u_\varepsilon) &= \lambda_k(u_0) + \varepsilon\, \nabla\lambda_k(u_0)\, r_k(u_0) + O(\varepsilon^2) \\ &= \lambda_k(u_0) + \varepsilon + O(\varepsilon^2) \\ &= \sigma(\varepsilon) + \frac{\varepsilon}{2} + O(\varepsilon^2) \end{aligned}$$

so that here again, imposing the lax inequality $\lambda_k(u(\varepsilon)) < \sigma(\varepsilon)$ leads to $\varepsilon < 0$. It now remains to prove that for $\varepsilon < 0$ sufficiently small we have $\lambda_{k-1}(u_0) < \sigma(\varepsilon) < \lambda_{k+1}(u(\varepsilon))$. In fact, we have

$$\lambda_{k-1}(u_0) < \lambda_k(u_0) < \lambda_{k+1}(u_0)$$

and we know that $\sigma(\varepsilon) \to \lambda_k(u_0)$ and $\lambda_{k+1}(u(\varepsilon)) \to \lambda_{k+1}(u_0)$ when ε goes to zero, so that the conclusion easily follows. ∎

Another way of selecting the physical part of the curve $\mathcal{S}_k(u_0)$ is naturally based on entropy considerations, as in the scalar case. Let us recall that a convex function $S: \Omega \to \mathbb{R}$ is called an entropy if there exists a smooth function $G: \Omega \to \mathbb{R}$, called entropy flux, such that $\nabla S(u)\, A(u) = \nabla G(u)$ for all $u \in \Omega$. Hence, using again the parametrization of lemma ?, we want to determine the states $\Psi_k(\varepsilon) \in \mathcal{S}_k(u_0)$ that satisfy the inequality

$$-\sigma(u_0, \Psi_k(\varepsilon))\big(S(\Psi_k(\varepsilon)) - S(u_0)\big) + \big(G(\Psi_k(\varepsilon)) - G(u_0)\big) \leq 0. \quad (36)$$

The next proposition proves that for <u>strictly convex entropies</u>, it is equivalent to impose (36) or the lax inequalities (33).

<u>Proposition</u>

Let (S, G) be an entropy pair. If the kth characteristic field is genuinely non linear and if S is strictly convex, then (36) holds true for $|\varepsilon|$ small enough if and only if $\varepsilon \leq 0$. (If the kth characteristic field is linearly degenerate, (36) is valid with an equality for all $|\varepsilon| \leq \varepsilon_1$)

<u>Proof</u> (sketch)

Let us set again $\sigma(\varepsilon) = \sigma(u_0, \Psi_k(\varepsilon))$ and $u(\varepsilon) = \Psi_k(\varepsilon)$, and define

$$E(\varepsilon) = \sigma(\varepsilon)\big(S(u(\varepsilon)) - S(u_0)\big) - \big(G(u(\varepsilon)) - G(u_0)\big).$$

The point is determine for which values of ε we have $E(\varepsilon) \geq 0$.

First, we note that $E(0)=0$. Next, differentiating gives

$$E'(\varepsilon) = \sigma'(\varepsilon)\left(S(u(\varepsilon)) - S(u_0)\right)$$
$$+\left(\sigma(\varepsilon)\nabla S(u(\varepsilon)) - \nabla S(u(\varepsilon))A(u(\varepsilon))\right)\cdot u'(\varepsilon)$$

Differentiating now the Rankine-Hugoniot relations

$$\sigma(\varepsilon)\left(u(\varepsilon) - u_0\right) = f(u(\varepsilon)) - f(u_0)$$

gives

$$\sigma'(\varepsilon)\left(u(\varepsilon) - u_0\right) = \left(A(u(\varepsilon)) - \sigma(\varepsilon)\,\mathrm{Id}\right)u'(\varepsilon).$$

Combining these two relations gives

$$E'(\varepsilon) = \sigma'(\varepsilon)\left(S(u(\varepsilon)) - S(u_0)\right)$$
$$- \nabla S(u(\varepsilon))\,\sigma'(\varepsilon)\left(u(\varepsilon) - u_0\right)$$
$$E'(\varepsilon) = \sigma'(\varepsilon)\left\{S(u(\varepsilon)) - S(u_0) - \nabla S(u(\varepsilon))\cdot(u(\varepsilon)-u_0)\right\}.$$

Then we have $E'(0)=0$. Differentiating again leads to $E''(0)=0$ and $E'''(0) = -\sigma'(0)\,\nabla^2 S(u_0)\,u'(0)\cdot u'(0)$.
If the kth characteristic field is genuinely nonlinear we thus have

$$E'''(0) = -\frac{1}{2}\,\nabla^2 S(u_0)\,r_k(u_0)\cdot r_k(u_0) < 0$$

by strict convexity. By a chain argument on the sign of the derivative and the monotonicity property, $\varepsilon \to E(\varepsilon)$ is then locally decreasing around $\varepsilon=0$. Thus, we must take $\varepsilon \leq 0$ to ensure $E(\varepsilon) \geq 0$.
Assume now that the kth characteristic field is linearly degenerate. Since we have $\sigma(\varepsilon) = \lambda_k(u(\varepsilon)) = \lambda_k(u_0)$, then $\sigma'(\varepsilon)=0$ so that $E'(\varepsilon)=0$. Then $E(0)=0$ implies $E(\varepsilon)$ for all ε which proves the validity of (36) with an equality. □

Solution to the Riemann problem

We are now in position to solve the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0) = \begin{cases} u_L & \text{if } x<0 \\ u_R & \text{if } x>0 \end{cases} \end{cases} \qquad (37)$$

We are going to state in this paragraph the main result of this theoretical part of the course, namely an existence and uniqueness result for sufficiently close initial states u_L and u_R (meaning that $\|u_L - u_R\|$ is small in some sense). Such a result is said to be local.

We begin by summarizing some of the results of the previous pages. Assume first that the kth characteristic field is genuinely nonlinear. Then we know, using the same notations, that the k-wave curve $\chi_k(u_0, \varepsilon)$ made of all the states that can be connected on the right to u_0 by either a rarefaction wave or an admissible shock wave is defined by

$$\chi_k(u_0, \varepsilon) = \begin{cases} \Phi_k(\varepsilon), & \varepsilon \geq 0 \\ \Psi_k(\varepsilon), & \varepsilon \leq 0 \end{cases} \qquad (38)$$

for $|\varepsilon|$ sufficiently small and is of class $\mathcal{C}^2$.

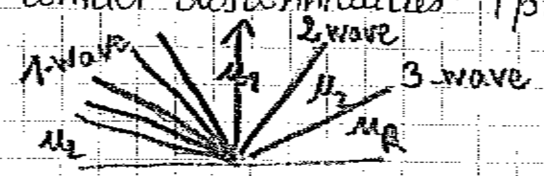
Assume now that the kth characteristic field is linearly degenerate. Then the k-wave curve $\chi_k(u_0, \varepsilon)$ made of all the states that can be connected to u_0 on the right by a contact discontinuity is defined by

$$\chi_k(u_0, \varepsilon) = \Psi_k(\varepsilon), \quad |\varepsilon| \text{ small enough} \qquad (39)$$

At last, we state the main result.

Theorem (Lax,)

Assume that for all $k = 1, \dots, p$, the kth characteristic field is either genuinely non linear or linearly degenerate. Then for all $u_L \in \Omega$ there exists a neighborhood $\mathcal{V}$ of u_L in Ω with the following property: if $u_R \in \mathcal{V}$, the Riemann problem (37) has a weak solution that consists of at most $(p+1)$ constant states separated by rarefaction waves, admissible shock waves, or contact discontinuities. Moreover, such a solution is unique.



Proof

Let us consider the mapping

$$\chi : \varepsilon = (\varepsilon_1, \dots, \varepsilon_p)^T \to \chi(\varepsilon) = \chi_p(\varepsilon_p, \chi_{p-1}(\varepsilon_{p-1}, \dots \chi_1(\varepsilon_1, u_L) \dots))$$

defined in a neighborhood of $0 \in \mathbb{R}^p$ and with values in Ω.
In other words, the left state is connected to $\chi_1(\varepsilon_1, u_L)$ by an admissible 1-wave, itself connected to $\chi_2(\varepsilon_2, \chi_1(\varepsilon_1, u_L))$ by an admissible 2-wave, and so on. We wonder wether the equation $\chi(\varepsilon) = u_R$ can be uniquely solved for any u_R in a neighborhood of u_L.
We first observe that $\chi(0) = u_L$.
Then, we know that for all k

$$\chi_k(\varepsilon_k, u) = u + \varepsilon_k r_k(u) + O(\varepsilon_k^2).$$

It thus comes

$$\begin{aligned}\chi_2(\varepsilon_2, \chi_1(\varepsilon_1, u_L)) &= \chi_2(\varepsilon_2, u_L + \varepsilon_1 r_1(u_L) + O(\varepsilon_1^2)) \\ &= u_L + \varepsilon_1 r_1(u_L) + O(\varepsilon_1^2) + \varepsilon_2 r_2(u_L + \varepsilon_1 r_1(u_L) + O(\varepsilon_1^2)) + O(\varepsilon_2^2) \\ &= u_L + \varepsilon_1 r_1(u_L) + \varepsilon_2 r_2(u_L) + O(\varepsilon_1^2 + \varepsilon_2^2)\end{aligned}$$

an by a chain argument

$$\chi(\varepsilon) = u_L + \sum_{k=1}^{p} \varepsilon_k r_k(u_L) + O(|\varepsilon|^2)$$

We then have $\nabla_\varepsilon \chi(0) = \begin{pmatrix} r_1 & \cdots & r_p \end{pmatrix}$. this matrix is invertible since the eigeinvectors are linearly independent. Then, we conclude by the local inversion theorem that there exists a neighborhood $\mathcal{V}$ of $u_L \in \Omega$ such that for all $u_R \in \mathcal{V}$, the equation $\chi(\varepsilon) = u_R$ has a unique solution consisting of $(p+1)$ constant states $u_0 = u_L$, $u_1, \dots, u_{p+1} = u_R$ separated by k-waves that are all admissible. ∎

Definition.

If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p)^t$ is the solution of $\chi(\varepsilon) = u_R$, then ε_k is called the strength of the kth wave in the solution of the Riemann problem (37).