The Cauchy-Lipschitz theorem ensures existence and uniqueness of a maximal solution on the interval $[\xi_0, \xi_{max}(v_0)[$, that we denote $v(\xi)$.
Assume in addition that v_0 and ξ_0 are such that

$$\lambda_k(v_0) = \xi_0 .$$

Then the function $\xi \to \lambda_k(v(\xi))$ coincides with $\xi \to \xi$ on $[\xi_0, \xi_{max}(v_0)[$ so that (19) is satisfied. Indeed, we easily get

$$\frac{d}{d\xi} \lambda_k(v(\xi)) = \nabla\lambda_k(v(\xi))\, v'(\xi) = \nabla\lambda_k(v(\xi))\, r_k(v(\xi)) = 1 .$$

so that $\lambda_k(v(\xi)) - \lambda_k(v(\xi_0)) = \xi - \xi_0$

ie $\lambda_k(v(\xi)) = \xi$.

For any $\xi_1 \in]\xi_0, \xi_{max}(v_0)[$, we say that $v_1 = v(\xi_1) \in \Omega$ is a state that can be joined on the right of v_0 by a k-rarefaction wave, (*) solution of (4)-(5) with $u_L = v_0$ and $u_R = v_1$. More precisely, the self-similar function

$$u(x,t) = \begin{cases} v_0 & \text{if } \frac{x}{t} \le \lambda_k(v_0) \\ v(\xi) & \text{if } \lambda_k(v_0) \le \frac{x}{t} = \xi \le \lambda_k(v_1) \\ v_1 & \text{if } \frac{x}{t} \geq \lambda_k(v_1) \end{cases}$$

satisfies (4)-(5) with

$$u(x,0) = \begin{cases} v_0 & \text{if } x < 0 \\ v_1 & \text{if } x > 0 . \end{cases}$$

(ie $u_L = v_0$ and $u_R = v_1$)

(*) We call $R_k(v_0)$ the set of admissible states that can be joined on the right of v_0 by a k-rarefaction wave, ie the set of all states $v(\xi)$, $\xi \in]\xi_0, \xi_{max}(v_0)[$, satisfying (20) with $\lambda_k(v_0) = \xi_0$.

We note sometimes

$$R_k(v_0) = \left\{ v \in \Omega \,/\, \begin{array}{l} v_0 \xrightarrow{k\text{-détente}} v \\ \lambda_k(v) \geq \lambda_k(v_0) \end{array} \right\}$$

Setting $\varepsilon = \xi - \lambda_k(v_0) \geq 0$ and $\phi_k(\varepsilon, v_0) = v(\xi)$, we have using a Taylor expansion

$$
\begin{aligned}
\phi_k(\varepsilon, v_0) &= v(\lambda_k(v_0) + \varepsilon) \\
&= v(\lambda_k(v_0)) + \varepsilon\, v'(\lambda_k(v_0)) + \frac{\varepsilon^2}{2} v''(\lambda_k(v_0)) + O(\varepsilon^3) \\
&= v(\xi_0) + \varepsilon\, v'(\xi_0) + \cancel{O(\varepsilon^2)} + \frac{\varepsilon^2}{2} v''(\xi_0) + O(\varepsilon^3) \\
&= v_0 + \varepsilon\, r_k(v_0) + \cancel{O(\varepsilon^2)} + \frac{\varepsilon^2}{2} \nabla r_k(v_0) . r_k(v_0) + O(\varepsilon^3)
\end{aligned}
$$

We have thus proved the following lemma

Lemma

Assume that the kth characteristic field is genuinely nonlinear with the normalization (16). Given $u_L \in \Omega$, there exists a curve $R_k(u_L) \subset \Omega$ that can be connected to u_L on the right by a k-rarefaction wave. Moreover, there exists a parametrization of $R_k(u_L)$: $\varepsilon \to \phi_k(\varepsilon, u_L)$ defined for $0 \leq \varepsilon \leq \varepsilon_0$, ε_0 small enough, such that

$$\phi_k(\varepsilon, u_L) = u_L + \varepsilon\, r_k(u_L) + \frac{\varepsilon^2}{2} \nabla r_k(u_L) . r_k(u_L) + O(\varepsilon^3). \quad (21)$$

We now conclude this paragraph by introducing the notion of Riemann invariants. Riemann invariants, as we shall see, allow to ~~pr~~ easily define the set $R_k(u_L)$ for all $u_L \in \Omega$ and are thus very useful in practice.

Definition

A smooth function $w: \Omega \subset \mathbb{R}^p \to \mathbb{R}$ is called a k-Riemann invariant if it satisfies

$$\nabla w(u) . r_k(u) = 0 \quad \forall u \in \Omega. \quad (22)$$

Remark

We readily note that when the kth field is linearly degenerate, λ_k is by definition a k-Riemann invariant since we have in this case $\nabla \lambda_k(u) . r_k(u) = 0$ (see the definition of a linearly degenerate field)

Actually, one can prove (see for instance the book by E. Godlewski and P.A. Raviart) that there exists locally around any $u_L \in \Omega$ $(p-1)$ k-Riemann invariants whose gradients are linearly independent. Then, a first interesting property is that a k-Riemann invariant is constant along the trajectories of the vector field r_k, that is along a k-rarefaction wave. Indeed we have using the same notations as before:

$$\begin{aligned} \frac{d}{d\xi} w(v(\xi)) &= \nabla w(v(\xi)) \cdot v'(\xi) \\ &= \nabla w(v(\xi)) \cdot r_k(v(\xi)) \\ &= 0 . \end{aligned}$$

In fact, it turns out that the $(p-1)$ k-Riemann invariants allow to parametrize the k-rarefaction curve. More precisely we can prove that $R_k(u_L) = \{ u \in \Omega$ such that $\lambda_k(u) \geq \lambda_k(u_L)$ and $w_i^k(u) = w_i^k(u_L), i = 1, p-1 \}$ where we have used the notations w_i^k to denote the $(p-1)$ Riemann invariants. We refer again to the book by E. Godlewski and P.A. Raviart for the proof of this result.

<u>Rankine-Hugoniot set</u>

After considering smooth solutions, we are now interested in discontinuous solutions of the form (17). More precisely, locally around such a discontinuity, it amounts to consider the self-similar solution

$$u(x,t) = \begin{cases} u_0 & \text{if } \frac{x}{t} < \sigma \\ u_1 & \text{if } \frac{x}{t} > \sigma \end{cases} \qquad (23)$$

where u_0 and u_1 denote two constant states in Ω and $\sigma \in \mathbb{R}$ is the speed of propagation of the discontinuity.

Let us first recall that (23) is a weak solution provided that it satisfies the Rankine-Hugoniot jump conditions

$$-\sigma(u_1 - u_0) + f(u_1) - f(u_0) = 0 \quad (24)$$

Given $u_0 \in \Omega$, our objective now is to determine all the states $u_1 \in \Omega$ to which u_0 can be connected on the right, that is such that (23) is valid for some speed of propagation σ. Then we introduce the following definition

<u>Definition</u>

The Rankine-Hugoniot set of $u_0 \in \Omega$ is the set of all states $u \in \Omega$ such that there exists $\sigma = \sigma(u_0, u) \in \mathbb{R}$ with

$$\sigma(u_0, u)(u - u_0) = f(u) - f(u_0) \qquad (25)$$

This set is denoted $\mathcal{S}(u_0)$.

Note that in the scalar case $p=1$, we clearly have $\mathcal{S}(u_0) = \Omega$ and $\sigma(u_0, u) \in \mathbb{R}$ is given by

$$\sigma(u_0, u) = \frac{f(u) - f(u_0)}{u - u_0}$$

as soon as $u \neq u_0$. When $u = u_0$, it is natural to set $\sigma(u_0, u_0) = f'(u_0)$.

In the general case of systems of conservation laws ($p \geq 1$), the structure of the Rankine-Hugoniot set of u_0 is given by the following lemma.

<u>Lemma</u>

Let u_0 be in Ω. The Rankine-Hugoniot set of u_0 is locally made of p smooth curves $\mathcal{S}_k(u_0)$, $k = 1, \dots, p$. For all k, there exists a parametrization of $\mathcal{S}_k(u_0)$: $\varepsilon \to \Psi_k(\varepsilon)$ defined for $|\varepsilon| \leq \varepsilon_1$, ε_1 small enough, such that

$$\Psi_k(\varepsilon) = u_0 + \varepsilon r_k(u_0) + \frac{\varepsilon^2}{2} \nabla r_k(u_0) \cdot r_k(u_0) + O(\varepsilon^3) \qquad (26)$$

and

$$\sigma(u_0, \Psi_k(\varepsilon)) = \lambda_k(u_0) + \frac{\varepsilon}{2} \nabla \lambda_k(u_0) \cdot r_k(u_0) + O(\varepsilon^2) \qquad (27)$$

<u>Remark</u>

Without restriction, we can consider the normalization (16) so that $\nabla \lambda_k(u_0) \cdot r_k(u_0)$ can be replaced by 1 in the second equality.

Proof

The first step of the proof consists in considering (25) as an eigenproblem. For that, it is sufficient to write

$$f(u)-f(u_0) = \int_0^1 \frac{d}{ds} f(u_0+s(u-u_0))\, ds$$

$$= \left(\int_0^1 \nabla f(u_0+s(u-u_0))\, ds\right)(u-u_0)$$

$$= \left(\int_0^1 A(u_0+s(u-u_0))\, ds\right)(u-u_0)$$

$$=: \mathcal{A}(u_0,u)(u-u_0),$$

so that (25) is equivalent to

$$\mathcal{A}(u_0,u)(u-u_0) = \sigma(u_0,u)(u-u_0). \quad (28)$$

Note that the $p\times p$ matrix $\mathcal{A}(u_0,u_0) = A(u_0)$ has p distinct eigenvalues $\lambda_k(u_0)$, and the function $u \to \mathcal{A}(u_0,u)$ is continuous. Thus, using a continuity argument, there exists a neighborhood $\mathcal{N}$ of u_0 in Ω and p real functions $u \to \lambda_k(u_0,u)$, $k=1,..,p$, defined in $\mathcal{N}$ and such that $\lambda_k(u_0,u)$, $k=1,..,p$, are the p distinct real eigenvalues of $\mathcal{A}(u_0,u)$ with $\lambda_k(u_0,u_0)=\lambda_k(u_0)$. Then (28) says that there exists k, $1\le k\le p$, such that

$$\begin{cases} \sigma(u_0,u) = \lambda_k(u_0,u) \\ (u-u_0) \text{ is colinear to } r_k(u_0,u) \end{cases}$$

or equivalently

$$\begin{cases} \sigma(u_0,u) = \lambda_k(u_0,u) & (i) \\ l_j(u_0,u)\cdot(u-u_0) = 0 \quad \forall j\neq k & (ii) \end{cases} \quad (29)$$

where $r_k(u_0,u)$ and $l_k(u_0,u)$ respectively denote the right, resp. left, eigenvectors of the matrix $\mathcal{A}(u_0,u)$.

For the sake of concision, we write the $(p-1)$ relations (29)(ii) for the p unknowns of u (its components) under the form

$$G_k(u) := M_k(u)(u-u_0) = 0 \quad (30)$$

with $M_k(u)$ the $(p-1)\times p$ matrix defined by

$$M_k(u) = \begin{pmatrix} l_1(u_0, u)^t \\ \vdots \\ l_{k_1}(u_0, u)^t \\ l_{k+1}(u_0, u)^t \\ \vdots \\ l_p(u_0, u)^t \end{pmatrix}$$

Roughly speaking, our objective is now to "invert" $M_k(u)$. Actually, since $G_k(u_0) = 0$ and $\nabla G_k(u_0) = M_k(u_0)$ is of maximal rank $(p-1)$ (the left eigenvectors $\{l_j(u_0, u)\}_{j\neq k}$ are linearly independent), the implicit function theorem allows to convert relations (30) to a one parameter function $\varepsilon \to \Psi_k(\varepsilon)$, $|\varepsilon| \leq \varepsilon_1$, ε_1 small enough (at least one component of u satisfying (30), say u^l can parametrize the other components $u^{i\neq l}$ in a neighborhood of $(u^l)_0$, but this neighborhood can be itself parametrized by ε, $|\varepsilon| \leq \varepsilon_1$, ε_1 small enough), satisfying $\Psi_k(0) = u_0$. Note also that $\sigma(u_0, \Psi_k(0)) = \sigma(u_0, u_0) = \lambda_k(u_0, u_0) = \lambda_k(u_0)$ by (29)(i) so that the term of order 0 in (26)(27) are proven.

Let us check now that

$$\Psi_k(\varepsilon) = u_0 + \varepsilon\, r_k(u_0) + O(\varepsilon^2)$$

Again, we start from (29)(ii) which writes

$$l_j(u_0, \Psi_k(\varepsilon)).(\Psi_k(\varepsilon) - \Psi_k(0)) = 0$$

then, we have for all $j \neq k$

$$\lim_{\varepsilon \to 0} l_j(u_0, \Psi_k(\varepsilon)) . \frac{(\Psi_k(\varepsilon) - \Psi_k(0))}{\varepsilon} = 0$$

ie $\qquad l_j(u_0, u_0) . \Psi_k'(0) = 0 \qquad \forall j \neq k,$

or $\qquad l_j(u_0) . \Psi_k'(0) = 0 \qquad \forall j \neq k,$

which proves that $\Psi_k'(0)$ is colinear to $r_k(u_0)$. Hence, we can change our parametrization in order to get $\Psi_k'(0) = r_k(u_0)$ as expected. The rest of the proof is not difficult but involves tedious calculations. It can be found in the book by E. Godlewski and P.-A. Raviart

Remark

It is worth noticing that thanks to (26) we have

$$\lambda_k(\Psi_k(\varepsilon)) = \lambda_k(u_0) + \varepsilon\, \nabla\lambda_k(u_0).r_k(u_0) + O(\varepsilon^2),$$

so that we get with (27)

$$\sigma(u_0, \Psi_k(\varepsilon)) = \frac{1}{2}\left(\lambda_k(u_0) + \lambda_k(\Psi_k(\varepsilon))\right) + O(\varepsilon^2).$$

The speed of propagation of the discontinuity (23) with $u_1 = \Psi_k(\varepsilon)$ can thus be approximated at second-order accuracy in ε by the mean value of $\lambda_k(u_0)$ and $\lambda_k(\Psi_k(\varepsilon))$ in the neighborhood of u_0.

Remark

Note that the parametrization of $\mathcal{R}_k(u_0)$ and $\mathcal{S}_k(u_0)$ admit the same Taylor expansions around u_0 up to the second-order in ε.

In order to go further into detail of the study of the Rankine-Hugoniot curves $\mathcal{S}_k(u_0)$, we now distinguish between a k-genuinely nonlinear characteristic field and a k-linearly degenerate characteristic field.

Definition

(i) If the characteristic field associated with λ_k is genuinely nonlinear then the curve $\mathcal{S}_k(u_0)$ is said to be a k-shock curve.

(ii) If the characteristic field associated with λ_k is linearly degenerate then the curve $\mathcal{S}_k(u_0)$ is said to be a k-contact discontinuity curve.