Definition

The system (1) is said to be symmetrizable on Ω if there exists a matrix $B_0(u)$, symmetric and positive definite $\forall u \in \Omega$, such that the matrices

$$B_j(u) = B_0(u)\, A_j(u), \quad \forall j = 1, \dots, d \qquad (8)$$

are symmetric. (or, equivalently, $B_j^{\omega}(u) = B_0 \sum_{j=1}^{d} \omega_j A_j$ is symmetric)
$\forall\, \omega = (\omega_1, \dots, \omega_d) \neq 0$

Lemma.

If the system (1) is symmetrizable on Ω, then it is hyperbolic on Ω.

Proof

The assumptions on B_0 allow to define with clear notations $B_0^{1/2} = P D_0^{1/2} P^t$ and $B_0^{-1/2} = P D_0^{-1/2} P^t$ where D_0 is a diagonal matrix with positive diagonal coefficients. Let us then consider the matrix $B_0^{-1/2} B_j^{\omega} B_0^{-1/2}$ for all $j = 1, \dots, d$. These matrices are clearly symmetric by assumption on B_j^{ω}, and then $\mathbb{R}$-diagonalizable. But these matrices are also similar to A_j^{ω} since

$$B_0^{-1/2} B_j^{\omega} B_0^{-1/2} = B_0^{1/2} A_j^{\omega} B_0^{-1/2}, \qquad A_j^{\omega} := \sum_{j=1}^{d} \omega_j A_j,$$

which easily leads to the expected conclusion ∎

The symmetrizability property is then stronger than the hyperbolicity property. Before stating the Leray-Schauder theorem, let us prove that like the hyperbolicity property, the symmetrizability property is invariant under change of variables (provided that it is symmetric).

Lemma

(Attention néanmoins : la notion de symétrie est [illegible] en variable [illegible] value u.)

The symmetrizability property is invariant under symmetric change of variables.

Proof

Let $u = v(u)$ be a symmetric change of variable ($u'(v)$ is symmetric). Then using the notations of Lemmas ? and ? it is first clear that $u' B_0 u' G_j^{\omega} = u' B_0 A_j^{\omega} u'$ is a symmetric matrix if $B_0 A_j^{\omega}$ is symmetric. In addition $\langle u' B_0 u' x, x\rangle = \langle B_0 u' x, u' x\rangle > 0 \ \forall x \in \mathbb{R}$ so that $u' B_0 u'$ is positive definite if B_0 does. This concludes the proof.

Let us now state the main result of this section.

<u>Theorem</u> (Leray-Schauder, 1934)
Assume that (1) is symmetric. Let u_0 be an initial condition in $H^s(\mathbb{R}^d)$ with values in K, a compact subset of Ω, with $s > 1 + d/2$ (in particular u_0 is a $\mathcal{C}^1$ function). Then there exists a time $T > 0$ such that the Cauchy problem (1)-(3) admits a smooth solution $u(x,t) \in \mathcal{C}^1([0,T] \times \mathbb{R}^d)$, unique in $\mathcal{C}([0,T], H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^d))$.

It is important to notice that T is generally finite (smoothness may be lost or $\partial\Omega$ reached in finite time).

The proof of this result is not given here and can be found in ?

To conclude this section, we now introduce the definition of entropy. We will see that if the system (1) admits a strictly convex entropy then it is symmetrizable (and then hyperbolic). Then, entropies may also be helpful to prove hyperbolicity. In addition, note from now on that entropies will be used in the forthcoming developments to single out a unique (weak) solution of (1)-(3), and more precisely of (4)-(5).

<u>Definition</u>
Assume that Ω is convex. A convex function $S : \Omega \to \mathbb{R}$ is called an (mathematical) entropy for (1) if there exist d functions $G_j : \Omega \to \mathbb{R}$, called the entropy fluxes, such that the relations

$$\nabla G_j(u) = \nabla S(u)\, A_j(u), \quad j = 1, \dots, d \quad (9)$$

hold true.

With this definition, assuming that u is a classical solution of (1) (or equivalently (2)) and carrying out a differentiation, we obtain that the additional conservation law

$$\partial_t S(u) + \sum_{j=1}^{d} \partial_{x_j} G_j(u) = 0$$

is satisfied.

Finding an entropy amounts to determining $(d+1)$ unknowns (G_j for j varying from 1 to d, and S) such that the $d \times p$ partial differential equations (9) are valid. In the scalar case $p=1$, we get an underdetermined system of equations: any convex function $S: \Omega \to \mathbb{R}$ is an entropy with entropy fluxes given by $G_j(u) = \int S'(u) f_j'(u)\, du$. In the general case $p>1$, finding entropies is much more complicated and even not always possible since we generally get an overdetermined set of equations. Note that the case $p=2, d=1$, which corresponds to the same numbers of unknowns and equations is treated in the book by D. Serre.
Fortunately, systems derived from mechanics and physics do admit an entropy in general.

Let us now see that a nonlinear system of conservation laws that admit a strictly convex entropy is symmetrizable. More precisely, we can state the following proposition.

<u>Proposition</u>:
Let $S: \Omega \to \mathbb{R}$ be a strictly convex function. A necessary and sufficient condition for S to be an entropy for (1) is that the $p \times p$ matrices $\nabla^2 S(u) A_j(u)$ are symmetric for $j=1,\dots,d$.

<u>Proof</u>
($\Rightarrow$) The relations $\nabla G_j = \nabla S\, A_j$ (or $\nabla G_j = \nabla S\, \nabla f_j$) write

$$\frac{\partial G_j}{\partial u_k} = \sum_{i=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \frac{\partial S}{\partial u_i}, \quad k=1,\dots,p$$

and give after differentiation

$$\frac{\partial^2 G_j}{\partial u_k \partial u_\ell} = \sum_{i=1}^{p} \frac{\partial^2 f_{ij}}{\partial u_k \partial u_\ell} \frac{\partial S}{\partial u_i} + \sum_{i=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \frac{\partial^2 S}{\partial u_i \partial u_\ell}.$$

sym in (k,ℓ)

Then, $(\nabla^2 S(u) A_j(u))_{k\ell} = (\nabla^2 S(u) \nabla f_j(u))_{k\ell} = \left(\sum_{i=1}^{p} \frac{\partial^2 S}{\partial u_i \partial u_k} \frac{\partial f_{ij}}{\partial u_\ell} \right)_{k\ell}$

is clearly symmetric.

($\Leftarrow$) Assume that $\nabla^2 S(u) A_j$ is symmetric. We want to prove the existence of entropy fluxes G_j such that $\nabla G_j = \nabla S \nabla f_j$, ie

$$\frac{\partial G_j}{\partial u_k} = \sum_{i=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \frac{\partial S}{\partial u_i}.$$

If we introduce the notation $T_k^j = \sum_{i=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \frac{\partial S}{\partial u_i}$, it is then sufficient to prove by Poincaré's theorem that

$$\frac{\partial T_k^j}{\partial u_\ell} = \frac{\partial T_\ell^j}{\partial u_k}.$$

We have $\frac{\partial T_k^j}{\partial u_\ell} = \sum_{i=1}^{p} \frac{\partial^2 f_{ij}}{\partial_k \partial u_\ell} \frac{\partial S}{\partial u_i} + \sum_{i=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \frac{\partial^2 S}{\partial u_i \partial u_\ell}$

which is clearly symmetric in (k,ℓ) ∎

Note first that an interest of Proposition ? lies in the fact that it does not involve the entropy fluxes.

In addition, and as an immediate corollary, the existence of a strictly convex entropy S implies that (1) is symmetrizable. Indeed, it suffices to take $\nabla^2 S$ (which is symmetric and positive-definite by strict convexity) since $\nabla^2 S A_j$ are symmetric for all j.

In other words, S allows to get the following symmetric form of system (1) in the u variable

$$\nabla^2 S(u)\, \partial_t u + \sum_{j=1}^{d} \nabla^2 S(u) A_j(u) \frac{\partial u}{\partial x_j} = 0.$$

Introducing the change of variable $v(u) = \nabla S(u)$, this equivalently writes

$$v'(u)\, \partial_t u + \sum_{j=1}^{d} v'(u) A_j(u)\, u'(v) \frac{\partial v}{\partial x_j} = 0$$

or (recall that $u'(v)\, v'(u) = id$)

$$u'(v)\, \partial_t v + \sum_{j=1}^{d} A_j(u)\, u'(v) \frac{\partial v}{\partial x_j} = 0 \quad (10)$$

where $u'(v) = \nabla^2 S(u)^{-1}$ is symmetric and positive-definite and $A_j(u)\, u'(v) = A_j(u)\, \nabla^2 S(u)^{-1} = \nabla^2 S(u)^{-1}\, \underline{\nabla^2 S(u)\, A_j}\, \nabla^2 S(u)^{-1}$ is symmetric.

We say that the change of variables $v(u)$ symmetrizes the system (1.1). (this is a definition)

Said differently, Proposition ? shows that if (1) admits a strictly convex entropy S, then there exists a change of variables $(v(u) = \nabla S(u))$ that symmetrizes the system (1.1). The next proposition proves that the reciproque is also true.

Proposition

A necessary and sufficient condition for (1) to possess a strictly convex entropy S is that there exists a change of variable $v(u)$ that symmetrizes (1). And we have $\nabla S(u) = v(u)$.

Proof

See the book by E. Godlewski and P.-A. Raviart. Not given here to avoid tedious calculations.

Exercise

Let us assume that the $p \times p$ matrices A_j in system (2) are symmetric. Show that the following function

$$S(u) = \frac{1}{2} \sum_{i=1}^{p} u_i^2, \quad u = (u_1, \dots, u_p)$$

is a strictly convex entropy with entropy fluxes given by

$$G_j(u) = \sum_{i=1}^{p} u_i f_{ij}(u) - g_j(u)$$

with $g_j = g_j(u)$ such that

$$\frac{\partial g_j}{\partial u_i} = f_{ij}$$

(also prove existence of this function).

Riemann Problem

Our objective is now to study the Riemann problem for a general nonlinear hyperbolic system of conservation laws in one space dimension. We first consider the linear case, and then address the nonlinear setting. Several notions will be introduced first in the latter case, and will play an important role hereafter: genuine nonlinearity, linear degeneracy, Riemann invariants, rarefaction waves, shock waves, contact discontinuities, entropy criterion .... Recall in particular that Riemann problems will be used as a building block to design relevant numerical schemes for approximating the physical solutions of (1).

Riemann problem for a strictly hyperbolic linear system with constant coefficients

We consider

$$\partial_t u + A \partial_x u = 0, \quad u \in \mathbb{R}^p, \quad A \in \mathcal{M}_{p\times p}(\mathbb{R}) \quad (1.1)$$

such that A admits the following distinct eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_p .$$

We denote r_k (respectively l_k^t) a right (resp. left) eigenvector associated with λ_k:

$$A r_k = \lambda_k r_k$$
$$(l_k^t A = \lambda_k l_k^t) \quad (1.2)$$

with the normalization

$$l_k^t r_k = 1 \quad (1.3)$$

Note that since the eigenvalues are distinct, the families $(r_k)_k$ and (l_k) form a basis of $\mathbb{R}^p$ and $l_i^t r_j = 0 \quad \forall i \neq j$: with usual notations we have indeed

$$(\lambda_j - \lambda_i)\, l_i^t r_j = (\lambda_j - \lambda_i)(l_i, r_j) = (l_i, \lambda_j r_j) - (\lambda_i l_i, r_j)$$
$$= (l_i, A\, r_j) - (A^t l_i, r_j)$$
$$= (A^t l_i, r_j) - (A^t l_i, r_j)$$
$$= 0$$

(then we necessarily have $l_i^t r_i \neq 0$ since $l_i \neq 0$ by definition)

The idea to solve the Riemann problem associated with (11) and the initial data

$$u_0(x) = u(x,0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

is simply to decompose u in the basis $(r_k)_k$:

$$u(x,t) = \sum_{k=1}^{p} \alpha_k(x,t)\, r_k$$

$$\text{with } \alpha_k(x,t) = (u(x,t), l_k)$$

Since we have

$$\partial_t u + A\,\partial_x u = \sum_{k=1}^{p} (\partial_t \alpha_k + \lambda_k \partial_x \alpha_k)\, r_k = 0$$

it is clear that for all $k = 1, \dots, p$:

$$\begin{cases} \partial_t \alpha_k + \lambda_k \partial_x \alpha_k = 0 \\ \alpha_k(x,0) = \alpha_k^0(x) = (u_0(x), l_k) \end{cases}$$

whose solution is given by $\alpha_k(x,t) = \alpha_k^0(x - \lambda_k t)$ so that

$$u(x,t) = \sum_{k=1}^{p} \alpha_k^0(x - \lambda_k t)\, r_k$$

$$\text{with } \alpha_k^0(x - \lambda_k t) = (u_0(x - \lambda_k t), l_k).$$

Then we have

- if $x < \lambda_1 t$ $\quad u(x,t) = \sum_{k=1}^{p} (u_L, l_k)\, r_k = u_L =: u_0$
- if $\lambda_1 t < x < \lambda_2 t$ $\quad u(x,t) = (u_R, l_1)\, r_1 + \sum_{k=2}^{p} (u_L, l_k)\, r_k =: u_1$

$\vdots$

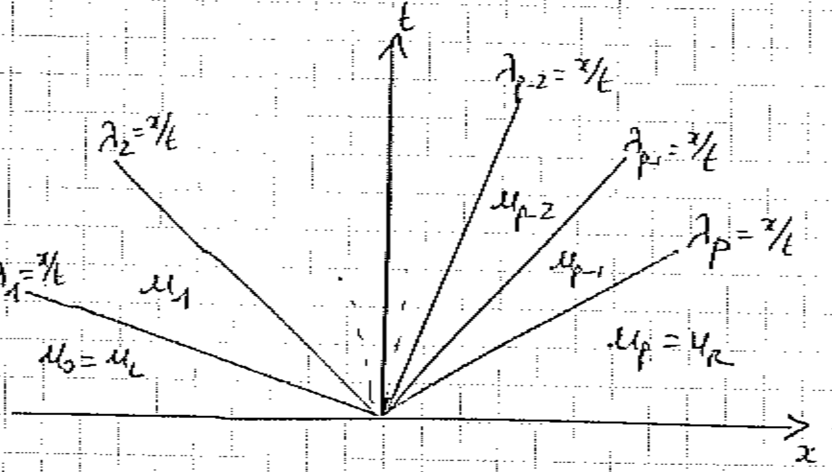
- if $x > \lambda_p t$ $\quad u(x,t) = \sum_{k=1}^{p} (u_R, l_k)\, r_k = u_R =: u_p$

The Riemann solution is then given explicitly by

$$u(x,t) \equiv u(x/t) = \begin{cases} u_0 = u_L & \text{if } \frac{x}{t} < \lambda_1 \\ u_1 & \text{if } \lambda_1 < \frac{x}{t} < \lambda_2 \\ u_2 & \text{if } \lambda_2 < \frac{x}{t} < \lambda_3 \\ \vdots & \\ u_p = u_R & \text{if } \lambda_p < x/t \end{cases}$$

with $\displaystyle u_i = \sum_{k=1}^{i} (u_R, l_k)\, r_k + \sum_{k=i+1}^{p} (u_L, l_k)\, r_k$

The wave pattern is as follows in the (x,t)-plane :



In general, the initial discontinuity then breaks up into p discontinuity waves which propagate with the characteristic speeds λ_k, $k=1,\dots,p$. Note in particular that

$$\begin{aligned} u_i - u_{i-1} &= (u_R, l_i)\, r_i - (u_L, l_i)\, r_i \\ &= \{(u_R, l_i) - (u_L, l_i)\}\, r_i \end{aligned}$$

so that

$$A(u_i - u_{i-1}) = \lambda_i (u_i - u_{i-1})$$

which means that the Rankine-Hugoniot relations are satisfied across each discontinuity.

We now turn to the non linear case and consider for simplicity the case of a strictly hyperbolic system with eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_p(u)$$

Again, the left and right eigenvector of $A(u)$ are denoted $l_k^t(u)$ and $r_k(u)$

Nature of the characteristic fields.

We first introduce the definitions of genuine nonlinearity and linear degeneracy

Definition.

(i) We say that the kth characteristic field is genuinely nonlinear if

$$\nabla\lambda_k(u).r_k(u) \neq 0 \quad \forall u \in \Omega \qquad (14)$$

(ii) We say that the kth characteristic field is linearly degenerate if

$$\nabla\lambda_k(u).r_k(u) = 0 \quad \forall u \in \Omega. \qquad (15)$$

It is important to have in mind that in the scalar case ($p=1$), we have $\lambda_1 = f'(u)$ so that genuine nonlinearity means that the flux function $f(u)$ is either strictly convex or strictly concave, while linear degeneracy means that $f(u)$ is an affine function (or let us say linear function since the flux function is defined up to a constant in $\partial_t u + \partial_x f(u) = 0$).

In the case of a genuinely nonlinear characteristic field, we will often choose to normalize the right eigenvector r_k so that

$$\nabla\lambda_k(u).r_k(u) = 1 \quad \forall u \in \Omega \quad (16)$$

(and the left eigenvector l_k^t still in compliance with (13)).

Like hyperbolicity, the above definition does not depend on the equivalent form under consideration of system (4).

Lemma

The genuine nonlinearity and linear degeneracy properties are invariant under change of variables.

Proof

We use the same notations as in the proof of Lemma ? :

$$\partial_t u + \partial_x f(u) = 0 \iff \partial_t u + A(u)\partial_x u = 0$$
$$\iff \partial_t v + C(v)\partial_x v = 0$$

with $C(v) = u'(v)^{-1} A(u(v))\, u'(v)$. Denote $\mu_k(v)$ and $s_k(v)$, $k=1,\dots,p$, the eigenvalues and the corresponding right eigenvectors of $C(v)$:

$$C(v)\, s_k(v) = \mu_k(v)\, s_k(v).$$

or equivalently

$$A(u(v))\, u'(v)\, s_k(v) = \mu_k(v)\, u'(v)\, s_k(v)$$

so that we can take

$$\begin{cases} \mu_k(v) = \lambda_k(u(v)), \\ s_k(v) = u'(v)^{-1} r_k(u(v)). \end{cases}$$

Then we have

$$\begin{aligned} \nabla \mu_k(v) \cdot s_k(v) &= u'(v)^t \nabla\lambda_k(u(v)) \cdot s_k(v) \\ &= u'(v)^t \nabla\lambda_k(u(v)) \cdot u'(v)^{-1} r_k(u(v)) \\ &= \nabla\lambda_k(u(v)) \cdot r_k(u(v)) \\ &= \nabla\lambda_k(u) \cdot r_k(u) \end{aligned}$$

with $u = u(v)$, which concludes the proof ∎

Rarefaction waves and Riemann invariants

Recall that we are interested in eventually solving the Riemann problem (4)-(5). Let us remark that if we denote by $u(x,t)$ a solution to (4)-(5), then for any $\lambda > 0$, the function $v^\lambda(x,t) = u(\lambda x, \lambda t)$ is again a solution since first

$$v^\lambda(x,0) = u(\lambda x, 0) = u_0(\lambda x) = u_0(x)$$

and then

$$\partial_t v^\lambda + \partial_x f(v^\lambda) = \lambda\left(\partial_t u + \partial_x f(u)\right) = 0.$$

Conjecturing the well-posedness of (4).(?), it is then natural to seek the solution under the self-similar form

$$u(x,t) = v(\xi), \quad \xi = x/t \qquad (17)$$

We focus in this paragraph on smooth solutions of this form, leading to the notion of rarefaction waves.

We first note if (17) obeys $\partial_t u + \partial_x f(u) = 0$, then we have

$$-\frac{x}{t^2} v'(\xi) + \frac{1}{t} A(v(\xi)) v'(\xi) = 0$$

$$\text{ie} \quad \big(A(v(\xi)) - \xi \, Id\big) v'(\xi) = 0. \quad (18)$$

Hence, there exists an index $k = 1, \dots, p$ such that

$$\begin{cases} v'(\xi) = \alpha(\xi) \, r_k(v(\xi)). \\ \lambda_k(v(\xi)) = \xi \end{cases} \qquad (19)$$

for all ξ (if $v'(\xi)$ is nonzero on an interval, and since the eigenvalues are distinct, a continuity argument ensures that k does not depend on ξ on this interval).

At this stage, it is very important to notice that differentiating the second relation in (19) gives

$$1 = \nabla \lambda_k(v(\xi)) \cdot v'(\xi)$$

and then using the first relation

$$1 = \nabla \lambda_k(v(\xi)) \cdot r_k(v(\xi)) \times \alpha(\xi)$$

As a conclusion, (19) cannot be solved for a linearly degenerate characteristic field. In other words, rarefaction waves make sens only for genuinely nonlinear characteristic fields.

Let us now consider the ordinary differential equation

$$\begin{cases} v'(\xi) = r_k(v(\xi)) \\ v(\xi_0) = v_0 \end{cases} \qquad (20)$$

(Under (16) we have $\alpha(\xi) = 1$)

with k associated with a genuinely nonlinear characteristic field and r_k such that (16) holds true.