Nonlinear hyperbolic systems of conservation laws.

This course is devoted to the theoretical and numerical study of nonlinear hyperbolic systems of conservation laws. Considering Ω an open subset of $\mathbb{R}^p$ and f_j, $1 \le j \le d$, d smooth functions from Ω into $\mathbb{R}^p$, the general form of such systems is

$$\partial_t u + \operatorname{div}_x f(u) = 0 \quad , \quad x = (x_1, \dots, x_d)^t \in \mathbb{R}^d, \ t > 0 \qquad (1)$$

or equivalently

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} f_j(u) = 0 \, , \quad x \in \mathbb{R}^d, \ t > 0 . \qquad (1)'$$

where $u = (u_1, \dots, u_p)^t \in \Omega \subset \mathbb{R}^p$ is the unknown and $f = (f_1, \dots, f_d)$ in $\mathbb{R}^{p \times d}$ is the flux function, with $f_j = (f_{1j}, \dots, f_{pj})^t \ \forall j = 1, \dots d$.
(1) and (1)' are called the <u>conservative</u> forms of the systems.

The chain rule also gives

$$\begin{aligned} \partial_t u + \operatorname{div}_x f(u) = 0 &\iff \partial_t u + \sum_{j=1}^{d} \partial_{x_j} f_j(u) = 0 \\ &\iff \partial_t u + \sum_{j=1}^{d} \nabla f_j \, \partial_{x_j} u = 0 \\ &\left(\iff \partial_t u_i + \sum_{j=1}^{d} \sum_{k=1}^{p} \frac{\partial f_{ij}}{\partial u_k} \times \frac{\partial u_k}{\partial x_j} = 0 \quad \forall i = 1, \dots, p \right) \end{aligned}$$

ie

$$\partial_t u + \sum_{j=1}^{d} A_j(u) \frac{\partial u}{\partial x_j} = 0 \qquad (2)$$

where we have set $A_j(u) = \nabla f_j(u) = \left(\dfrac{\partial f_{ij}}{\partial u_k} \right)_{(i,k)} \in \mathbb{R}^{p \times p}$.

(2) is called a <u>nonconservative</u>(*) form of the system.
System (1) is supplemented with an initial condition

$$u(x, 0) = u_0(x) \quad , \quad x \in \mathbb{R}^d \qquad (3)$$

where $u_0 : \mathbb{R}^d \to \Omega$ is a given function.

(*) Note that except if $p = 1$, a nonconservative form does not necessarily admit an equivalent conservative form.

In this course, a particular attention will be paid to the one-dimensional case, namely

$$\partial_t u + \partial_x f(u) = 0 \quad , \underline{x \in \mathbb{R}}, \ t > 0, \quad (4)$$

and to the so-called Riemann initial conditions with the following particular form:

$$u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases} \qquad (5)$$

where u_L and u_R are two constant states in Ω.

(4)-(5) is called the (one-dimensional) <u>Riemann problem</u>.

<u>Definition</u>.

(i) The system (1) is said to be hyperbolic on Ω if for any $u \in \Omega$, and any $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$, $\omega \neq 0$, the matrix $A(u,\omega) = \sum_{j=1}^{d} \omega_j A_j(u)$ is diagonalizable in $\mathbb{R}$, that is if this matrix has p real eigenvalues $\lambda_1(u,\omega) \leq \dots \leq \lambda_p(u,\omega)$ and p linearly independant corresponding eigenvectors $r_1(u,\omega), \dots, r_p(u,\omega)$.

(ii) The system (1) is said to be strictly hyperbolic if the eigenvalues are all distinct.

<u>Why is hyperbolicity important?</u>

Let us first recall that a mathematical problem is well-posed in the sense of Hadamard if it has the properties that

- a solution exists
- the solution is unique
- the solution depends continuously on the data, in some reasonable topology ($\| u_2 - u_1 \| \leq C \| u_2^0 - u_1^0 \|$)

Actually, hyperbolicity is a necessary condition for (1)-(3) to be well-posed. For the purpose of illustration, let us consider a linear system in 1D :

$$\partial_t u + A\,\partial_x u = 0 \ , \ x \in \mathbb{R},\ t \geq 0 \qquad (6)$$

cas linéaire : stabilité (*) $\|u(t,x)\| \leq C\,\|u(0,x)\|$ $\forall t$

where A is a constant matrix in $\mathbb{R}^{p\times p}$, with for instance $p=2$.

→ Let us first consider the case $A = P\begin{pmatrix}\lambda & 0\\ 0 & \bar\lambda\end{pmatrix}P^{-1}$ with $\lambda = a+ib \neq \bar\lambda = a-ib$ (ie $b \neq 0$), which corresponds to a $\mathbb{C}$-diagonalizable matrix, but not to a $\mathbb{R}$-diagonalizable matrix. Assume without restriction that $b>0$, and denote $\bar q$ an eigenvector associated with $\bar\lambda$: $A\bar q = \bar\lambda \bar q$.
Take now $u_0(x) = e^{-ikx}\,\bar q$. The solution of (6) is then given by $u(x,t) = e^{bkt}\,e^{i(akt-kx)}\,\bar q$ (check that $\partial_t u = (bk+iak)\,u$ and $A\partial_x u = -ki\,(a-ib)\,u$). It is clear that the amplitude of e^{bkt} grows up with k (or with t), while u_0 is clearly bounded in L^2. Then, the stability is lost (utterly)

→ Let us now assume that $A = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ which has real eigenvalues but is not diagonalizable. Then, (6) writes

$$\begin{cases} \partial_t u_1 + \lambda\,\partial_x u_1 = -\,\partial_x u_2 \,, \\ \partial_t u_2 + \lambda\,\partial_x u_2 = 0 \,, \end{cases}$$

the solution of which is given by

$$\begin{cases} u_1(x,t) = u_1^0(x-\lambda t) - t\, u_2^{0\,\prime}(x-\lambda t) \\ u_2(x,t) = u_2^0(x-\lambda t) \end{cases} \qquad (7)$$

where the superscript "o" refers to the initial condition. Here, the stability is lost again but in a weaker sense since it holds true provided that the initial condition is given a more restrictive norm (with one more derivative), which is not satisfactory(*)

→ At last, if we assume that A is $\mathbb{R}$-diagonalizable : $A = PDP^{-1}$ with $D = \mathrm{diag}(\lambda_k)$, then setting $v = P^{-1}u$ gives $\partial_t v + D\,\partial_x v = 0$.
The p equations of this equivalent form are clearly decoupled and the solution is given by $v(x,t) = (v_1, \dots, v_p)(x,t)$ with $v_k(x,t) = v_k^0(x-\lambda_k t)$ $\forall k = 1,\dots,p$. In this case, (6) is stable in the Hadamart sense.

(*) We also note that (7) is a measure-solution as soon as the initial condition is discontinuous

The following property is useful to show the hyperbolicity property in practice.

Lemma

The hyperbolicity property is invariant under change of variables.

Proof

Let $u \to v(u)$, or equivalently $v \to u(v)$, be an admissible change of variables. We have by (2) and with clear notations

$$u'(v)\partial_t v + \sum_{j=1}^{d} A_j(u(v))\, u'(v) \frac{\partial v}{\partial x_j} = 0$$

that is $\partial_t v + \sum_{j=1}^{d} C_j(v) \frac{\partial v}{\partial x_j} = 0$ with $C_j(v) = u'(v)^{-1} A_j(u(v))\, u'(v)$.

The matrices $C_j(v)$ and $A_j(u(v))$ being similar, it is clear that the matrices $C(v,\omega) = \sum_{j=1}^{d} \omega_j C_j(v)$ and $A(u,\omega) = \sum_{j=1}^{d} \omega_j A_j(u)$ share the same $\mathbb{R}$-diagonalizability property. This concludes the proof ∎

In practice, it is often time-saving to first find a relevant change of variables to prove the hyperbolicity property and find the eingenvalues and eigenvectors.

Let us now introduce the definition of symmetrizable system, the interest of which is twofold. First, a symmetrizable system will be shown to be hyperbolic, and then it necessarily admits a smooth solution for small times by the well-known Leray-Schauder theorem.