Interface model coupling via prescribed local flux balance

Thomas Galié * and Annalisa Ambroso †
CEA, Gif-sur-Yvette, 91191, France
Christophe Chalons ‡
Université Paris Diderot, Paris, F-75005, France
Frédéric Coquel §
Université Pierre et Marie Curie-Paris6, Paris, F-75005, France

This paper presents two methods for coupling two different models. Each of these models is based on the equations of gas dynamics in Eulerian coordinates for a one-dimensional barotropic gas. The two models are separated on the domain by a thin interface and differ for their equation of state. In order to control the coupling conditions, we choose to add a local source term in the system describing the complete model.

I. Introduction

In many industrial applications, the mathematical description of complex systems often needs to be divided in few sub-systems either to simplify the problem or to reduce computation times. This is the case in nuclear applications for example, where the coolant flow in the primary circuit can be described by different models in the components of the circuit. Then one may have to couple these codes in order to get a simulation tool for the whole system. To allow for the coupling of codes, the idea is to look for compatible boundary conditions on the interface between the models.

The problem of interface coupling of systems in fluid dynamics has recently received attention. In[9] and[8] the authors study theoretical and numerical methods to couple two hyperbolic systems of conservation laws at a fixed interface. In[9] it is shown that we cannot preserve the continuity of the solution at the interface and impose conservativity of the coupled model at the same time. In[1] the authors pointed out that a uniform profile cannot always be preserved if the conservativity of the coupled model is respected. On the other hand, they developed some new numerical techniques in order to couple two gas dynamic equation systems for a one dimensional polytropic ideal gas in Eulerian coordinates. The two systems have different adiabatic coefficient so that the momentum flux and the energy flux are discontinuous at the interface. The methods given in,[1] as expected, broke out at least the conservation of energy if a uniform profile for the pressure is imposed. In[11] a system that models a free medium is coupled with a porous one. A study of the coupling of multidimensional systems can be found in[12] where one 1D-model is coupled with a 2D-model. Finally, we can refer the reader to[2] , [4] and[12] for examples of coupling between multiphase flow models.

In this paper, we analyze the problem of the coupling of two models for the flow of a barotropic gas in one space dimension. The PDE's describing the flow are the same, but the equations of state on each subdomain are different. We propose to model the coupling conditions by a local source term concentrated at the interface. The model is presented in section II. In section III, we describe two methods to compute the solution of the coupled model: the first one is based on the resolution of two Riemann problems and

*PhD student, Centre de Saclay, DM2S/SFME/LETR.
†Research scientist, Centre de Saclay, DM2S/SFME/LETR.
‡Assistant professor, Université Paris Diderot and UMR 7598 Laboratoire Jacques-Louis Lions.
§CNRS researcher, Université Pierre et Marie Curie-Paris6, UMR 7598 Laboratoire Jacques-Louis Lions.

the second one uses a relaxation model approach. In section IV, we compare numerical results for these two methods.

II. Presentation of the coupling problem

II.A. Statement of the PDE problem

We consider the flow of a barotropic gas in one dimension. We denote $x \in \mathbb{R}$ the space variable whereas $t > 0$ denotes time. The space domain $\mathbb{R}$ is separated by a thin interface $\mathcal{I}$ located at $x = 0$. The system is described on each side of the interface by the equations of gas dynamics but with different equations of state so that the pressure law $p(\tau)$, where τ is the specific volume, differs across the interface and shifts from $p_L(\tau)$ for $x < 0$ to $p_R(\tau)$ for $x > 0$. Thus, we write the coupling problem as:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0, \qquad t > 0, \quad x < 0, \tag{1}$$

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0, \qquad t > 0, \quad x > 0, \tag{2}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \qquad \mathbf{f}_\alpha(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p_\alpha(\tau) \end{pmatrix} \quad \text{for} \quad \alpha = L, R, \quad \text{and} \quad \tau = \frac{1}{\rho}$$

and with the initial conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}. \tag{3}$$

The unknowns are the density of the fluid ρ and the velocity of the fluid u. Pressure laws $p_\alpha(\tau)$ with $\alpha = L, R$ are assumed to obey $p'_\alpha(\tau) < 0$ and $p''_\alpha(\tau) > 0$ for all $\tau > 0$: namely the left and right systems in Eqs. (1) and (2) are strictly hyperbolic over the following natural phase space:

$$\omega = \{\mathbf{u} = (\rho, \rho u) \in \mathbb{R}^2, \rho > 0, \rho u \in \mathbb{R}\}. \tag{4}$$

The characteristic fields of the two systems are genuinely nonlinear and each one has two eigenvalues respectively given by

$$\lambda^{\pm}_\alpha(\mathbf{u}) = u \pm c_\alpha(\tau), \quad c_\alpha(\tau) = \tau\sqrt{-p'_\alpha(\tau)}, \quad \alpha = L, R. \tag{5}$$

If we write our coupling problem in a condensed form, we get, provided that $x \neq 0$:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, x) = 0, \quad t > 0, \quad x \in \mathbb{R}\backslash\{0\}, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}. \end{cases}, \quad \mathbf{f}(\mathbf{u}, x) = \begin{cases} \mathbf{f}_L(\mathbf{u}), & x < 0, \\ \mathbf{f}_R(\mathbf{u}), & x > 0. \end{cases} \tag{6}$$

Notice that no information is given on $x = 0$. The whole problem of coupling consists in defining these conditions. Finding coupling conditions has been studied in[1] for the coupling of two gas dynamic equations for a polytropic ideal gas with discontinuous pressure law. In this work, the authors give different methods to couple the two systems. One, by a relaxation approach with a so-called color function,[1] computes one flux at the interface and the result is the strictly conservativity of the method. The other coupling method[1] is based on the resolution of two independent problems on the left and on the right. Two boundary conditions have to be set and two fluxes at the interface calculated. This method is no longer conservative as the two fluxes are not in general equal, but it allows to compute solutions with uniform pressure profiles.

Motivated by these observations, we propose in this paper a new approach which consists in a global problem where the coupling conditions are given by imposing a measure valued load on the interface. By measure valued we mean that the load is concentrated on $x = 0$. The coupling model take the following form:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, x) = \mathcal{M}(t)\delta_{x=0}, \quad t > 0, \quad x \in \mathbb{R}, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{7}$$

where $\delta_{x=0}$ is the Dirac delta function in $x = 0$. The required definition of the time dependant weight $\mathcal{M}(t)$ actually models the coupling conditions to be prescribed at the interface $x = 0$ in System (6). Indeed, being given a suitable definition of the weight $\mathcal{M}(t)$, solving the coupling Problem (7) amounts to find a solution $\mathbf{u}(x,t)$, $t > 0$, $x \in \mathbb{R}\backslash\{0\}$ of System (6) subject to the following coupling conditions at $x = 0$:

$$\mathbf{f}_R(\mathbf{u}(0^+,t)) - \mathbf{f}_L(\mathbf{u}(0^-,t)) = \mathcal{M}(t), \quad t > 0, \tag{8}$$

where $\mathbf{u}(0^+,t)$ and $\mathbf{u}(0^-,t)$ respectively denote the left and right trace at the interface $x = 0$ of the solution $\mathbf{u}$ at time t. For instance, considering the case of a conservative coupling (i.e. $\mathbf{f}_R(\mathbf{u}(0^+,t)) = \mathbf{f}_L(\mathbf{u}(0^-,t))$), one must obviously require the weight $\mathcal{M}(t)$ to identically vanish. By contrast, general transmission conditions[3] result in a non conservative coupling and thus give birth to non zero weights $\mathcal{M}(t)$. In the present work, we focuse ourselves on weights under the form:

$$\mathcal{M}(t) = \begin{pmatrix} \mathcal{M}_\rho(t) \\ \mathcal{M}_{\rho u}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{M}_{\rho u}(t) \end{pmatrix}, \quad t > 0, \tag{9}$$

in order to preserve the conservation of the density ρ which is a fundamental physical property in our applications. Such problems have been studied in[6] and[10] for example. The pressure drop law $\mathcal{M}_{\rho u}(t)$ must be prescribed according to the Physics. Typical values of $\mathcal{M}_{\rho u}(t)$ are discussed in[7] and their influence on the flow.

The main goal of the present work is to derive efficient numerical methods for the approximation of the solution of the coupling Problem (7). Away from the interface, this problem boils down to a well-known conservative problem. Here the difficulty is to enforce for validity the coupling condition (8).
With this in mind, we will require the exact capture of the equilibrium solutions introduced in the next definition.

Definition. *Let $\mathcal{M}$ be a constant weight and $\mathbf{u}_L$, $\mathbf{u}_R$ two constant states in ω such that*

$$\mathbf{f}_R(\mathbf{u}_R) - \mathbf{f}_L(\mathbf{u}_L) = \mathcal{M}, \tag{10}$$

then the function

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_L, & x < 0, \quad t > 0, \\ \mathbf{u}_R, & x > 0, \quad t > 0, \end{cases}$$

is called an equilibrium solution for Problem (7).

II.B. General numerical standpoint

In a first time, let us introduce the numerical notations. We note Δt the time step, Δx the space step and $\nu = \Delta t/\Delta x$ their ratio. The interfaces of the cells are in $x_j = j\Delta x$ for $j \in \mathbb{Z}$ and intermediate times are $t^n = n\Delta t$ for $n \in \mathbb{N}$. We now seek at each time t^n a piecewise constant approximate solution $x \to \mathbf{u}_\nu(x,t^n)$ of the solution $\mathbf{u}$ of Problem (7):

$$\mathbf{u}_\nu(x,t^n) = \mathbf{u}^n_{j+1/2} \quad \text{for} \quad x \in C_{j+1/2} = [x_j, x_{j+1}).$$

At $t = 0$ we set from (3)

$$\mathbf{u}^0_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \mathbf{u}_0(x)dx, \ j \in \mathbb{Z}.$$

We use, without restriction, a 3-point finite-volume method to compute the updating of the discrete solution $\mathbf{u}_\nu(x,t^n)$:

$$\mathbf{u}^{n+1}_{j-1/2} = \mathbf{u}^n_{j-1/2} - \nu((\mathbf{g}_L)^n_j - (\mathbf{g}_L)^n_{j-1}), \qquad j \le 0, \tag{11}$$

$$\mathbf{u}^{n+1}_{j+1/2} = \mathbf{u}^n_{j+1/2} - \nu((\mathbf{g}_R)^n_{j+1} - (\mathbf{g}_R)^n_j), \qquad j \ge 0, \tag{12}$$

where:

$$(\mathbf{g}_L)_j^n = \mathbf{g}_L(\mathbf{u}_{j-1/2}^n, \mathbf{u}_{j+1/2}^n),$$
$$(\mathbf{g}_R)_j^n = \mathbf{g}_R(\mathbf{u}_{j-1/2}^n, \mathbf{u}_{j+1/2}^n).$$

The numerical fluxes, $\mathbf{g}_L$ and $\mathbf{g}_R$, are respectively two Riemann approximations (obtained for instance from two approximate Riemann solvers) of the physical fluxes $\mathbf{f}_L$ and $\mathbf{f}_R$. The coefficient ν is chosen by a classical CFL restriction. The numerical flux $(\mathbf{g}_L)_0^n$ (respectively $(\mathbf{g}_R)_0^n$) corresponds to the exact flux $\mathbf{f}_L(\mathbf{u}(0^-,t))$ (resp. $\mathbf{f}_R(\mathbf{u}(0^+,t))$) at the interface. Their definition must be related to the prescribed weight $\mathcal{M}(t)$ and they will be derived in order to provide a good approximation of the coupling Condition (8) at each time t^n:

$$(\mathbf{g}_R)_0^n - (\mathbf{g}_L)_0^n \approxeq \mathcal{M}^n, \quad \mathcal{M}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathcal{M}(t)dt, \quad n > 0. \tag{13}$$

In the next section, we describe two methods to compute the numerical fluxes $(\mathbf{g}_{L,R})_0^n$ in a way to obtain Eq. (13).

III. Methods for coupling

III.A. Two-Riemann problem method

The first method we present consists in solving two distinct Riemann problems, respectively labelled by L and R at the interface $x = 0$ so as to define the numerical fluxes $(\mathbf{g}_L)_0^n$ and $(\mathbf{g}_R)_0^n$.
In order to define $(\mathbf{g}_L)_0^n$, we propose to handle the weight in the approximate Formula (13) when defining a ghost state $\mathbf{u}^L(\mathbf{u}_{1/2}^n, \mathcal{M}^n)$ as the solution of:

$$\mathbf{f}_R(\mathbf{u}_{1/2}^n) - \mathbf{f}_L(\mathbf{u}^L(\mathbf{u}_{1/2}^n, \mathcal{M}^n)) = \mathcal{M}^n. \tag{14}$$

Equipped with such a solution, we define:

$$(\mathbf{g}_L)_0^n = \mathbf{g}_L(\mathbf{u}_{-1/2}^n, \mathbf{u}^L(\mathbf{u}_{1/2}^n, \mathcal{M}^n)). \tag{15}$$

In a symmetric way, to define $(\mathbf{g}_R)_0^n$, the weight $\mathcal{M}^n$ is addressed when solving for the ghost state $\mathbf{u}^R(\mathbf{u}_{-1/2}^n, \mathcal{M}^n)$ the next system of equations:

$$\mathbf{f}_R(\mathbf{u}^R(\mathbf{u}_{-1/2}^n, \mathcal{M}^n)) - \mathbf{f}_L(\mathbf{u}_{-1/2}^n) = \mathcal{M}^n, \tag{16}$$

so as to introduce

$$(\mathbf{g}_R)_0^n = \mathbf{g}_R(\mathbf{u}^R(\mathbf{u}_{-1/2}^n, \mathcal{M}^n), \mathbf{u}_{1/2}^n). \tag{17}$$

Due to the strict convexity of each of the pressure law p_α, $\alpha = L, R$, it can be seen that the nonlinear algebraic Problems (14) and (16) to be solved admit either zero, or two solutions depending on the amplitude of $||\mathcal{M}^n||$, the states $\mathbf{u}_{-1/2}^n$ and $\mathbf{u}_{1/2}^n$ being fixed. In practice, the left and right fluxes $\mathbf{f}_L$ and $\mathbf{f}_R$ do not depart too much so that expected values of the weight generically yield two solutions for each problem: namely a subsonic and a supersonic solution.
Motivated by our multiphase flow applications, we always choose the subsonic solutions. The following statement assesses the validity of the so-called two-Riemann problem method.

Lemma 1. *Let be given two constant states $\mathbf{u}_-$ and $\mathbf{u}_+$ in ω and a constant weight $\mathcal{M}$ such that:*

$$\mathbf{f}_R(\mathbf{u}_+) - \mathbf{f}_L(\mathbf{u}_-) = \mathcal{M}. \tag{18}$$

Then the solution $\mathbf{u}^L(\mathbf{u}_R, \mathcal{M})$ of Eq. (14) (respectively $\mathbf{u}^R(\mathbf{u}_L, \mathcal{M})$ of Eq. (16)) exists and yields the next identities:

$$\mathbf{g}_L(\mathbf{u}_-, \mathbf{u}^L(\mathbf{u}_+, \mathcal{M})) = \mathbf{f}_L(\mathbf{u}_-),$$
$$\mathbf{g}_R(\mathbf{u}^R(\mathbf{u}_-, \mathcal{M}), \mathbf{u}_+) = \mathbf{f}_R(\mathbf{u}_+).$$

The two-Riemann problem method (14), (15) and (16), (17) thus preserves stationnary solutions of the coupling Problem (7).

Proof. At first we evaluate the states $\mathbf{u}^L$ and $\mathbf{u}^R$; they are the unique subsonic solutions of

$$\mathbf{f}_R(\mathbf{u}^+) - \mathbf{f}_L(\mathbf{u}^L(\mathbf{u}^+, \mathcal{M})) = \mathcal{M},$$
$$\mathbf{f}_R(\mathbf{u}^R(\mathbf{u}^-, \mathcal{M})) - \mathbf{f}_L(\mathbf{u}^-) = \mathcal{M}.$$

By using Eq. (18), we have

$$\mathbf{f}_L(\mathbf{u}^L(\mathbf{u}^+, \mathcal{M})) = \mathbf{f}_L(\mathbf{u}^-),$$
$$\mathbf{f}_R(\mathbf{u}^R(\mathbf{u}^-, \mathcal{M})) = \mathbf{f}_R(\mathbf{u}^+).$$

□

III.B. A Relaxation model approach

In this section, we propose a coupling method based on a Relaxation approach for the approximation of the hyperbolic system (7). We refer the reader to[5] and[14] for general discussions on the Relaxation approach in the approximation of hyperbolic systems of conservation laws. Our discussion is divided in two subsections. At first, we will show how to solve the Riemann problem for the Relaxation model associated to the coupling problem in the case of the conservative coupling model (i.e. when $\mathcal{M}(t)$ completely vanishes in Problem (7)). Then, the second subsection will treat the case of the non-conservative coupling problem (i.e. with $\mathcal{M}(t) \neq 0$) that is the approximation by the Relaxation model of Problem (7).

III.B.1. Conservative coupling by the Relaxation method

We begin this section by writing the coupling Cauchy Problem (6) closed by condition $\mathcal{M}(t) = 0$, with initial data $\mathbf{u}_0(x)$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\tau, x)) = 0, \quad t > 0, \quad x \in \mathbb{R} \end{cases} \tag{19}$$

where

$$p(\tau, x) = \begin{cases} p_L(\tau), & x < 0, \\ p_R(\tau), & x > 0. \end{cases}$$

The consequence of System (19) is the equality of the physical fluxes evaluated on the traces of the interface:

$$\rho u(0^-, t) = \rho u(0^+, t), \tag{20}$$
$$(\rho u^2 + p_L(\tau))(0^-, t) = (\rho u^2 + p_R(\tau))(0^+, t). \tag{21}$$

We propose to approximate the solutions of Problem (6) by means of the following Relaxation system

$$\begin{cases} \partial_t \rho_\lambda + \partial_x(\rho u)_\lambda = 0, \\ \partial_t(\rho u)_\lambda + \partial_x(\rho u^2 + \pi)_\lambda = 0, \\ \partial_t(\rho \pi)_\lambda + \partial_x(\rho \pi u + a^2 u)_\lambda = \lambda \rho_\lambda (p(\tau, x) - \pi)_\lambda, \quad t > 0, \quad x \in \mathbb{R}, \end{cases} \tag{22}$$

where a is a free parameter that we will discuss later on. The relaxation approach consists in smoothing the strong nonlinearities of the initial system by replacing it with a new system. In this model, the pressure law is replaced by a new unknown noted here π_λ. Observe that the solutions of the original model are formally restored when considering infinite values of the relaxation parameter $\lambda > 0$, since we have formally

$$\lim_{\lambda \to +\infty} \pi_\lambda = p(\tau, x),$$

from the last PDE in System (22). However it is known[14] that to prevent the relaxation procedure from instabilities in the regime of large $\lambda >> 1$, the free parameter $a > 0$ in System (22) has to be chosen so as to meet

$$a^2 > \max_\tau(-p_L'(\tau), -p_R'(\tau)) \tag{23}$$

for all τ under consideration. This condition is called the *Whitham condition*. It guarantees the stability of the Relaxation model. The Eqs. (20) and (21) become in the relaxation model:

$$(\rho u)_\lambda(0^-,t) = (\rho u)_\lambda(0^+,t),$$
$$(\rho u^2+\pi)_\lambda(0^-,t) = (\rho u^2+\pi)_\lambda(0^+,t).$$

We now write System (22) in a condensed form like for Problem (7) (for the sake of simplicity we decide not to write the subscript λ in the following):

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \lambda \mathbf{S}(\mathbf{U},x), \quad x\in\mathbb{R}, \quad t>0, \tag{24}$$

with

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho\pi \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2+\pi \\ \rho\pi u + a^2 u \end{pmatrix}, \quad \mathbf{S}(\mathbf{U},x) = \begin{pmatrix} 0 \\ 0 \\ \rho(p(\tau,x)-\pi) \end{pmatrix}.$$

We now define the new space Ω of states $\mathbf{U}$:

$$\Omega = \{\mathbf{U}=(\rho,\rho u,\rho\pi)\in\mathbb{R}^3, \rho>0, \rho u\in\mathbb{R}, \rho\pi\in\mathbb{R}\}. \tag{25}$$

System (24) is strictly hyperbolic for all $\mathbf{U}\in\Omega$ and for all a. We describe now the numerical resolution of Problem (19) by means of System (22). Consider the discrete solution $\mathbf{u}_\nu(x,t^n)$ of Problem (19) at time t^n. Let us remind that it is a piecewise constant function taking constant values on each cell $C^n_{j+1/2}$. The evolution of this solution at time t^{n+1} is made in two steps.

First step: time evolution ($t^n \to t^{n+1-}$)

During the first step, we want to solve for $t\in[0,\Delta t]$, Δt small enough,

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0, \quad x\in\mathbb{R}, \quad t>0, \\ \mathbf{U}(x,0) = \mathbf{U}_\nu(x,t^n), \end{cases} \tag{26}$$

with

$$\mathbf{U}_\nu(x,t^n) = (\mathbf{u}_\nu(x,t^n),(\rho\pi)_\nu(x,t^n))^T \tag{27}$$

and where $\rho\pi$ is at equilibrium that is we have

$$(\rho\pi)_\nu(x,t^n) = \rho(x,t^n)p(\tau_\nu(x,t^n),x).$$

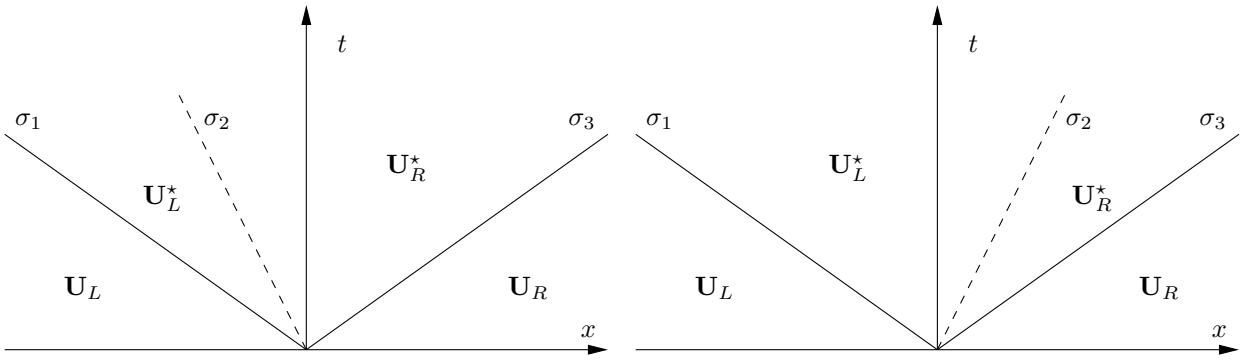
Under an appropriate CFL restriction, the solution of (26) is a succession of Riemann problems. The system has three increasingly arranged eigenvalues:

$$\lambda_1(\mathbf{U}) = u - a\tau, \quad \lambda_2(\mathbf{U}) = u, \quad \lambda_3(\mathbf{U}) = u + a\tau. \tag{28}$$

The fields of this system are all linearly degenerate, so that the solution of the Riemann problem contains only contact discontinuities. More precisely, consider $\mathbf{U}_L$ and $\mathbf{U}_R$ two values of successive cells and solve

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0, \quad x\in\mathbb{R}, \quad t>0, \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L, & x<0, \\ \mathbf{U}_R, & x>0. \end{cases} \end{cases} \tag{29}$$

The structure of the solution for Problem (29) is easy to get (see Figure (1)): this self-similar solution we denote $\mathcal{W}(x/t;\mathbf{U}_L,\mathbf{U}_R)$ is composed of four constant states $\mathbf{U}_L$, $\mathbf{U}^\star_L$, $\mathbf{U}^\star_R$ and $\mathbf{U}_R$ systematically separated by discontinuities with velocity σ_i, $i=\{1,2,3\}$. The contact discontinuities are moving with the velocity σ_i



(a) Case $\sigma_2 < 0$ (b) Case $\sigma_2 > 0$

Figure 1. Solution of the Riemann problem for the conservative coupling relaxation model

of each field so we have:

$$\sigma_1 = \lambda_1(\mathbf{U}_L) = \lambda_1(\mathbf{U}_L^\star),$$
$$\sigma_2 = \lambda_2(\mathbf{U}_L^\star) = \lambda_2(\mathbf{U}_R^\star),$$
$$\sigma_3 = \lambda_3(\mathbf{U}_R^\star) = \lambda_3(\mathbf{U}_R).$$

Using now the Rankine-Hugoniot relations and the equations above leads easily[5] to the expressions of the intermediate states $\mathbf{U}_L^\star$ and $\mathbf{U}_R^\star$. They are obtained from:

$$\tau_L^\star = \tau_L + \frac{1}{a}(u^\star - u_L), \tag{30}$$

$$\tau_R^\star = \tau_R + \frac{1}{a}(u_R - u^\star), \tag{31}$$

$$u^\star = u_L^\star = u_R^\star = \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(\pi_R - \pi_L), \tag{32}$$

$$\pi^\star = \pi_L^\star = \pi_R^\star = \frac{1}{2}(\pi_L + \pi_R) - \frac{a}{2}(u_R - u_L). \tag{33}$$

The updating of the discrete states is determined by using a classical three-point Godunov method. More precisely, the numerical flux at each interface is equal to the physical flux of the Relaxation model evaluated on the left (or right) trace given by the resolution of the Riemann problem. One can notice by the Relations (30)-(33), that the intermediate states do not contain the pressure law inhomogeneity. This one only appears in the initial condition (27) and we have

$$\mathbf{F}(\mathbf{U}(0^-, t)) = \mathbf{F}(\mathbf{U}(0^+, t)),$$

where $\mathbf{U}(0^-, t)$ and $\mathbf{U}(0^+, t)$ are the traces of the solution at the interface. We thus have the continuity of the fluxes for the relaxation method.

Second step: relaxation $(t^{n+1-} \to t^{n+1})$

The solution obtained after this convective step is noted $\mathbf{U}_\nu(x, t^{n+1-})$. In the second step, we solve the following system of ordinary differential equations

$$\begin{cases} \partial_t \rho = 0 \\ \partial_t(\rho u) = 0 \\ \partial_t(\rho\pi) = \lambda(\rho - \pi), \end{cases}$$

with $\lambda \to +\infty$ and initial condition given by the solution of the first step $\mathbf{U}_\nu(x, t^{n+1-})$. Finally the discrete solution $\mathbf{u}_\nu(x, t^{n+1})$ corresponds to

$$\mathbf{U}_\nu(x, t^{n+1}) = (\mathbf{u}_\nu(x, t^{n+1}), (\rho\pi)_\nu(x, t^{n+1}))^T$$

with

$$\begin{aligned}\mathbf{u}_\nu(x,t^{n+1}) =& \mathbf{u}_\nu(x,t^{n+1-})\\ (\rho\pi)_\nu(x,t^{n+1}) =& \rho(x,t^{n+1-})p(\tau_\nu(x,t^{n+1-}),x).\end{aligned}$$

To summarize this paragraph, we give the detailed form of the numerical fluxes $(\mathbf{g}_L)_0^n$ and $(\mathbf{g}_R)_0^n$ entering Eqs. (11) and (12). At time t^n, being given the states $\mathbf{u}_{-1/2}^n$ and $\mathbf{u}_{1/2}^n$, we define $\mathbf{U}_{-1/2}^n = (\mathbf{u}_{-1/2}^n, \rho_{-1/2}^n p_L(\tau_{-1/2}^n))$ and $\mathbf{U}_{1/2}^n = (\mathbf{u}_{1/2}^n, \rho_{1/2}^n p_R(\tau_{1/2}^n))$ according to Eq. (27). The required fluxes $(\mathbf{g}_L)_0^n$ and $(\mathbf{g}_R)_0^n$ are recovered from the first two components of $\mathbf{F}(\mathcal{W}(0^+;\mathbf{U}_{-1/2}^n,\mathbf{U}_{1/2}^n))$ setting

$$(\mathbf{g}_L)_0^n = (\mathbf{g}_R)_0^n = \begin{pmatrix} \rho u \\ \rho u^2+\pi \end{pmatrix}\left(\mathcal{W}(0^+;\mathbf{U}_{-1/2}^n,\mathbf{U}_{1/2}^n)\right). \tag{34}$$

III.B.2. Non conservative coupling by the Relaxation method

The presence of the local source term $\mathcal{M}\delta_{x=0}$ in Problem (7) changes the resolution of the Riemann problem in the Relaxation system. In order to keep the same numerical method, based on the Relaxation model and the corresponding Godunov method to evaluate the flux on the interface, we have to express the new intermediate states of the solution of the Riemann problem. These states, and in particular the traces on the left and right of the interface depend explicitly on the measure load. We write now the new Riemann problem (29) with the local source term:

$$\begin{cases} \partial_t\mathbf{U}+\partial_x\mathbf{F}(\mathbf{U}) = \widetilde{\mathcal{M}}(t)\delta_{x=0}, \quad x\in\mathbb{R}, \quad t>0,\\ \mathbf{U}(x,0) = \begin{cases}\mathbf{U}_L, & x<0,\\ \mathbf{U}_R, & x>0.\end{cases}\end{cases} \tag{35}$$

where $\widetilde{\mathcal{M}}(t)$ is built from $\mathcal{M}=(0,\mathcal{M}_{\rho u}(t))$ in Problem (7) when setting

$$\widetilde{\mathcal{M}}(t) = \begin{pmatrix} \mathcal{M}(t) \\ \mathcal{M}_{\rho\pi}(t)\end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{M}_{\rho u}(t) \\ \mathcal{M}_{\rho\pi}(t)\end{pmatrix}. \tag{36}$$

Here, the additionnal weight $\mathcal{M}_{\rho\pi}$ acting on the last equation of System (22) seems arbitrary but actually must be given a precise non zero definition for the exact capture of equilibrium solutions of the original coupling Problem (7) (see Proposition (1) below).
Let us briefly comment the structure of the self-similar solution of the relaxation coupling Problem (35). In comparison with the last paragraph, an additionnal wave with zero speed - the so-called standing wave - has to be dealt with. This supplementary wave accounts for the source term $\widetilde{\mathcal{M}}(t)$ concentrated at $x=0$ and yields the traces of the solution of System (35) to obey

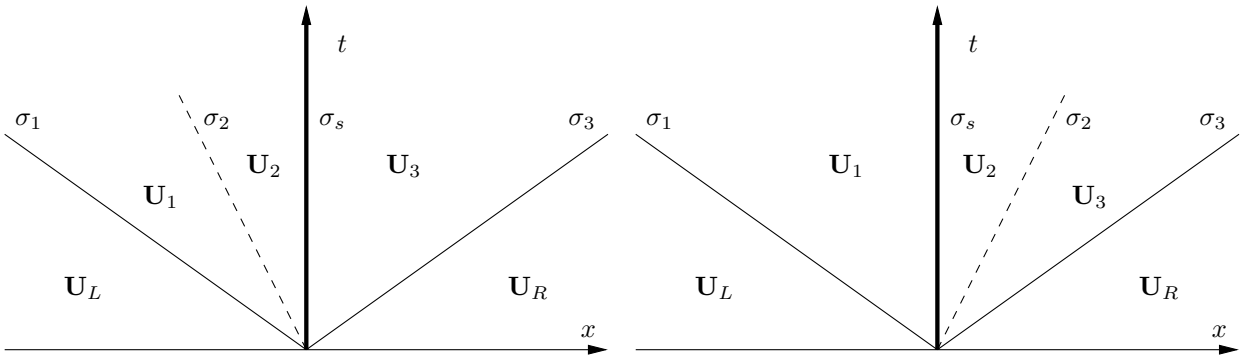
$$(\rho u)_+ - (\rho u)_- = 0, \tag{37}$$

$$(\rho u^2+\pi)_+ - (\rho u^2+\pi)_- = \mathcal{M}_{\rho u}, \tag{38}$$

$$(\rho\pi u + a^2 u)_+ - (\rho\pi u + a^2 u)_- = \mathcal{M}_{\rho\pi}. \tag{39}$$

In agreement with our multiphase flow motivation, we will only address the case of fully subsonic self-similar solutions of System (35): namely the left and right traces of the Riemann solutions are subsonic. Hence, the typical pattern of the self-similar solutions under consideration is displayed in Figure (2). Observe that the expected self-similar solution of System (35) does depend not only on the given data $\mathbf{U}_L$, $\mathbf{U}_R$ but also on the two prescribed weights $\mathcal{M}_{\rho u}$ and $\mathcal{M}_{\rho\pi}$. The left and right states $\mathbf{U}_L$ and $\mathbf{U}_R$ being fixed, arbitrary values of the weights may result in a wave pattern distinct from the one depicted in Figure (2). Their values must thus be conveniently monitored in order to fit with this expected waves structure. In that aim, it turns to be convenient to reexpress the weight $\mathcal{M}_{\rho\pi}$ as a function of the weight $\mathcal{M}_{\rho u}$ and the mass flux:

$$m = \rho_- u_- = \rho_+ u_+, \tag{40}$$



(a) Case $\sigma_2 < 0$ (b) Case $\sigma_2 > 0$

Figure 2. Solution of the coupled Riemann problem with the local term $\mathcal{M}(t)$ for the convection system of the relaxation model

at the interface:

$$\mathcal{M}_{\rho\pi} = \mathcal{M}_{\rho\pi}(m, \mathcal{M}_{\rho u}; \mathbf{U}_L, \mathbf{U}_R), \tag{41}$$

which we will write $\mathcal{M}_{\rho\pi}(m, \mathcal{M}_{\rho u})$ for short. The next statement[7] is then in order:

Proposition 1. *Let be given two subsonic states $\mathbf{U}_L$ and $\mathbf{U}_R$ in Ω^2, then there exists a non empty bounded convex domain $\mathcal{D}_{adm}(\mathbf{U}_L, \mathbf{U}_R) \subset \mathbb{R}_m \times \mathbb{R}_{\mathcal{M}_{\rho u}}$ such that for any given pair $(m, \mathcal{M}_{\rho u}) \in \mathcal{D}_{adm}(\mathbf{U}_L, \mathbf{U}_R)$ there exists an unique self-similar subsonic solution we denote $\mathcal{W}(x/t; m, \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ of Problem (35) with weight $\widetilde{\mathcal{M}} = (0, \mathcal{M}_{\rho u}, \mathcal{M}_{\rho\pi}(m, \mathcal{M}_{\rho u}))$.*

In this paper, we only pay attention to specific definitions of $\widetilde{\mathcal{M}}$ in Eq. (36) from the given $\mathcal{M}$ in Problem (7) such that equilibrium solutions of Problem (7) are exactly restored by the relaxation procedure. More precisely, two states $\mathbf{u}_L$ and $\mathbf{u}_R$ in ω being given as a solution of:

$$\mathbf{f}_R(\mathbf{u}_R) - \mathbf{f}_L(\mathbf{u}_L) = \mathcal{M}, \tag{42}$$

for some $\mathcal{M} = (0, \mathcal{M}_{\rho u})$, $\mathcal{M}_{\rho u} \in \mathbb{R}$, then define $\mathbf{U}_L = (\mathbf{u}_L, \rho_L p_L(\tau_L))$ and $\mathbf{U}_R = (\mathbf{u}_R, \rho_R p_R(\tau_R))$, the mass flux m in Eq. (40) must be properly defined such that the solution $\mathcal{W}(x/t; m, \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ of Problem (35) writes:

$$\mathcal{W}(x/t; m, \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{U}_L, & x < 0, \quad t > 0, \\ \mathbf{U}_R, & x > 0. \end{cases} \tag{43}$$

In other words, the initial data of Problem (35) is nothing but the solution. Restricting such a solution to its two first components restores the next function

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_L, & x < 0, \quad t > 0, \\ \mathbf{u}_R, & x > 0. \end{cases} \tag{44}$$

This function is just the expected equilibrium solution of Problem (7). The next statement provides the relevant definition of the required mass flux m:

Proposition 2. *Let be given $(\mathbf{u}_L,\ \mathbf{u}_R)$ in ω^2 and $\mathcal{M}_{\rho u} \in \mathbb{R}$ such that Eq. (42) holds with $\mathcal{M} = (0, \mathcal{M}_{\rho u})$. Let us define:*

$$m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R) = \frac{\mathcal{M}_{\rho u} + 2a u^\star(\mathbf{U}_L, \mathbf{U}_R)}{a(\tau_L^\star(\mathbf{U}_L, \mathbf{U}_R) + \tau_R^\star(\mathbf{U}_L, \mathbf{U}_R))}, \tag{45}$$

where the mapping $\tau_L^\star$, $\tau_R^\star$ and $u^\star$ have been defined in Eqs. (30), (31) and (32). Then the solution $\mathcal{W}(x/t; m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R), \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ coincides with the initial data $\mathbf{U}_0(x) = \mathbf{U}_L = (\mathbf{u}_L, \rho_L p_L(\tau_L))$ if $x < 0$ and $\mathbf{U}_R = (\mathbf{u}_R, \rho_R p_R(\tau_R))$ if $x > 0$. The definition (45) of the mass flux m thus preserves the equilibrium solutions of Problem (7).

We conclude when detailing the definition of the three intermediate states $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ involved in the waves patterns depicted in Figure (2). In this aim, it is convenient to consider the left and right traces of the self-similar solution $\mathcal{W}(x/t; m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R), \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ singled out in the above statement at the interface, which we denote respectively $\mathbf{U}_- = \mathcal{W}(0^-; m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R), \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ and $\mathbf{U}_+ = \mathcal{W}(0^+; m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R), \mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$. Observe that $m_e \equiv m_e(\mathcal{M}_{\rho u}, \mathbf{U}_L, \mathbf{U}_R)$ (which we write $m_e(\mathcal{M}_{\rho u})$ for short) in (45) verifies $m_e < 0$ as soon as $\sigma_2 < 0$ and $m_e > 0$ whenever $\sigma_2 > 0$. With these solutions, observe that in Figure (2(a)) with $\sigma_2 < 0$, we have $\mathbf{U}_- = \mathbf{U}_2$ and $\mathbf{U}_+ = \mathbf{U}_3$ while by contrast in Figure (2(b)) with $\sigma_2 > 0$, we have $\mathbf{U}_- = \mathbf{U}_1$ and $\mathbf{U}_+ = \mathbf{U}_2$. Equipped with the definition of the mass flux (45), let us introduce

$$\mathcal{M}^e_{\rho\pi}(m_e(\mathcal{M}_{\rho u}), \mathbf{U}_L, \mathbf{U}_R) = m_e(\mathcal{M}_{\rho u})(\mathcal{I}_R - \mathcal{I}_L), \tag{46}$$

where we have set $\mathcal{I}_R = p_R(\tau_R) + a^2\tau_R$ and $\mathcal{I}_L = p_L(\tau_L) + a^2\tau_L$ which we write for short $\mathcal{M}^e_{\rho\pi}(\mathcal{M}_{\rho u})$. Then the left and right traces $\mathbf{U}_-$ and $\mathbf{U}_+$ at $x = 0$ of the self-similar solution displayed in Figure (2) are recovered from:

$$u_- = u^\star + \frac{a\mathcal{M}_{\rho u} - \mathcal{M}^e_{\rho\pi}(\mathcal{M}_{\rho u})}{2a(a - m_e(\mathcal{M}_{\rho u}))}, \qquad \tau_- = \frac{u_-}{m_e(\mathcal{M}_{\rho u})}, \qquad \pi_- = \pi^\star - \frac{2a\mathcal{M}_{\rho u} - \mathcal{M}^e_{\rho\pi}(\mathcal{M}_{\rho u})}{2(a - m_e(\mathcal{M}_{\rho u}))}, \tag{47}$$

$$u_+ = u^\star + \frac{a\mathcal{M}_{\rho u} + \mathcal{M}^e_{\rho\pi}(\mathcal{M}_{\rho u})}{2a(a + m_e(\mathcal{M}_{\rho u}))}, \qquad \tau_+ = \frac{u_+}{m_e(\mathcal{M}_{\rho u})}, \qquad \pi_+ = \pi^\star + \frac{2a\mathcal{M}_{\rho u} + \mathcal{M}^e_{\rho\pi}(\mathcal{M}_{\rho u})}{2(a + m_e(\mathcal{M}_{\rho u}))}, \tag{48}$$

as soon as $m_e(\mathcal{M}_{\rho u}) \neq 0$. If $m_e(\mathcal{M}_{\rho u})$ vanishes, then $\mathbf{U}_- = \mathbf{U}_+ = \mathbf{U}^\star_L$ if $u^\star > 0$ and $\mathbf{U}_- = \mathbf{U}_+ = \mathbf{U}^\star_R$ otherwise (see Figure (2). Then to conclude, we need to define $\mathbf{U}_3$ in the case $m_e(\mathcal{M}_{\rho u}) > 0$ (i.e. $\mathbf{U}_1 = \mathbf{U}_-, \mathbf{U}_2 = \mathbf{U}_+$) by:

$$u_3 = u_+, \quad \tau_3 = \tau_R + \frac{1}{a}(u_R - u_+), \quad \pi_3 = \pi_+, \tag{49}$$

and $\mathbf{U}_1$ in the case $m_e(\mathcal{M}_{\rho u}) < 0$ (i.e. $\mathbf{U}_2 = \mathbf{U}_-, \mathbf{U}_3 = \mathbf{U}_+$) by:

$$u_1 = u_-, \quad \tau_1 = \tau_L - \frac{1}{a}(u_L - u_-), \quad \pi_1 = \pi_-. \tag{50}$$

To summarize the expected numerical fluxes entering Eqs. (11) and (12) are recovered as follow. At time t^n, being given the states $\mathbf{u}^n_{-1/2}$ and $\mathbf{u}^n_{1/2}$, we define $\mathbf{U}^n_{-1/2} = (\mathbf{u}^n_{-1/2}, \rho^n_{-1/2} p_L(\tau^n_{-1/2}))$ and $\mathbf{U}^n_{1/2} = (\mathbf{u}^n_{1/2}, \rho^n_{1/2} p_R(\tau^n_{1/2}))$ according to Eq. (27). Then from the value of $\mathcal{M}^n = (0, \mathcal{M}^n_{\rho u} = \mathcal{M}_{\rho u}(t^n))$ in (7) we define $\widetilde{\mathcal{M}}^n = (0, \mathcal{M}^n_{\rho u}, \mathcal{M}^e_{\rho\pi}(\mathcal{M}^n_{\rho u}))$ with $\mathcal{M}^e_{\rho\pi}(\mathcal{M}^n_{\rho u})$ in (46). The solution $\mathcal{W}(x/t; m_e(\mathcal{M}^n_{\rho u}), \mathcal{M}^n_{\rho u}, \mathbf{U}^n_{-1/2}, \mathbf{U}^n_{1/2})$ of (35) being deduced from the formulae (47) to (48) we have:

$$(\mathbf{g}_L)^n_0 = \begin{pmatrix} \rho u \\ \rho u^2 + \pi \end{pmatrix} \left(\mathcal{W}(0^-; m_e(\mathcal{M}^n_{\rho u}), \mathcal{M}^n_{\rho u}, \mathbf{U}^n_{-1/2}, \mathbf{U}^n_{1/2}) \right), \tag{51}$$

$$(\mathbf{g}_R)^n_0 = \begin{pmatrix} \rho u \\ \rho u^2 + \pi \end{pmatrix} \left(\mathcal{W}(0^+; m_e(\mathcal{M}^n_{\rho u}), \mathcal{M}^n_{\rho u}, \mathbf{U}^n_{-1/2}, \mathbf{U}^n_{1/2}) \right). \tag{52}$$

IV. Results

We present in this section results from numerical simulations based on the methods we propose. In the next test cases, two distinct cases of values for $\mathcal{M}_{\rho u}$ are considered. The initial condition for the flow are of Riemann type. The two constant states $\mathbf{u}_L$ and $\mathbf{u}_R$ under consideration are given by:

$$\mathbf{u}(x, 0) = \begin{cases} \mathbf{u}_L & \text{if} \quad x < 0, \\ \mathbf{u}_R & \text{if} \quad x > 0, \end{cases}$$

with

	L	R
ρ	2	1
u	u_L	0

where

$$u_L = u_R + \sqrt{(p_R(\tau_R) - p_L(\tau_L))(\tau_L - \tau_R)}, \quad \tau_R < \tau_L.$$

The pressure law is $p_\alpha(\tau) = \tau^{-\gamma_\alpha}$. The adiabatic coefficients are $\gamma_L = 1.4$ for the left domain and $\gamma_R = 1.6$ for the right domain. The number of mesh points is 200 and the space domain is the interval $[-0.5, 0.5]$. The CFL constant is set to 0.5.

IV.A. Conservative coupling $\mathcal{M} \equiv (0, 0)$

The purpose of the present benchmark is to illustrate the behavior of the numerical methods we have introduced to approximate the solution of Problem (7) in the conservative setting: i.e. with $\mathcal{M} = (0, 0)$. We show that the two-Riemann problem method and the relaxation strategy described in Section (III) yield discrete solutions in a fairly good agreement despite some minor discrepencies may be reported. Discrete solutions are compared on Figures (3) and (4). We observe that the approximate solution obtained by the two-Riemann problem approach exhibits a perfectly sharp discontinuity at the interface while the relaxation approach described in Section (III) displays a discrete profile with one intermediate point resulting in a slight overshoot in the pressure distribution. To go further, Figures (8(a)) and (9(a)) display the component of impulsion of difference $(\mathbf{g}_R)_0^n - (\mathbf{g}_L)_0^n$ in its time history and the value $\mathcal{M}_{\rho u}^n$ (that is here strictly zero). As expected from its design principle, the relaxation approach yields an exact balance $(\mathbf{g}_R)_0^n - (\mathbf{g}_L)_0^n = 0$ at each time step. The two-Riemann problem approach achieves this exact balance after a few time steps but departure from the expected cancellation stays fairly admissible.

Figure (5) shows the results achieved by the relaxation approach described in Section (III.B.2) with $\mathcal{M}_{\rho\pi}^e(\mathcal{M}_{\rho u})$ given in (46) choosing $\mathcal{M}_{\rho u} \equiv 0$. Notice that the present method differs from the first relaxation strategy described in Section (III.B.1) since $\mathcal{M}_{\rho\pi}^e(0) \neq 0$ in general. The benefit of this second approach clearly stays in the capture of a perfectly sharp standing wave at the interface. Let us stress that this second method exactly restores by construction the identity $(\mathbf{g}_R)_0^n = (\mathbf{g}_L)_0^n$.

IV.B. Non conservative coupling

In this test case, the value of $\mathcal{M}_{\rho u}$ is chosen so as to enforce the continuity of the traces of the exact Riemann solution in Problem (7) at $x = 0$; namely $\mathbf{u}(0^-, \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}(0^+, \mathbf{u}_L, \mathbf{u}_R)$. It can be seen[7] that the expected value is given by:

$$\mathcal{M}_{\rho u}^n = 0.354404, \quad n > 0.$$

Figures (6) and (7) compare the discrete solutions obtained by the two-Riemann problem approach and the relaxation strategy. The discrete standing waves are quite comparable. Then Figures (8(b)) and (9(b)) display the component of impulsion of the difference $(\mathbf{g}_R)_0^n - (\mathbf{g}_L)_0^n$ and the exact value of $\mathcal{M}_{\rho u}^n$ used here. Again by construction such a difference is strictly equal to the prescribed weight at all time steps in the relaxation strategy while it takes few time steps in the two-Riemann problem method.

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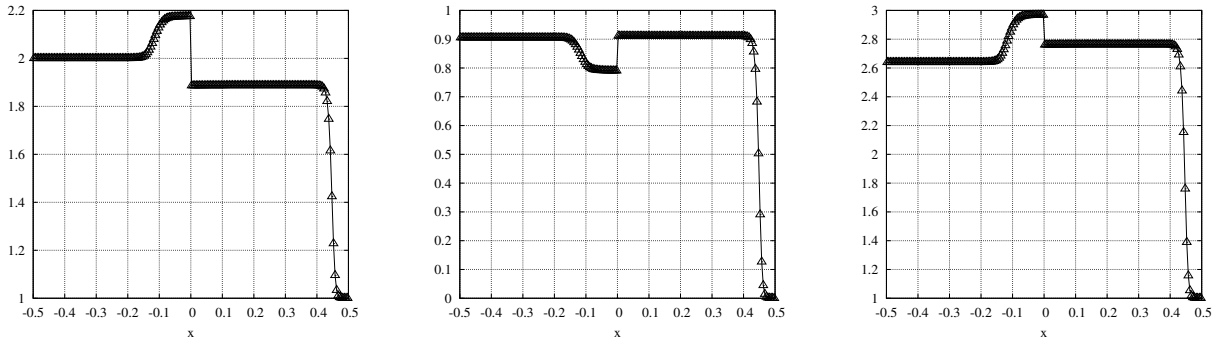
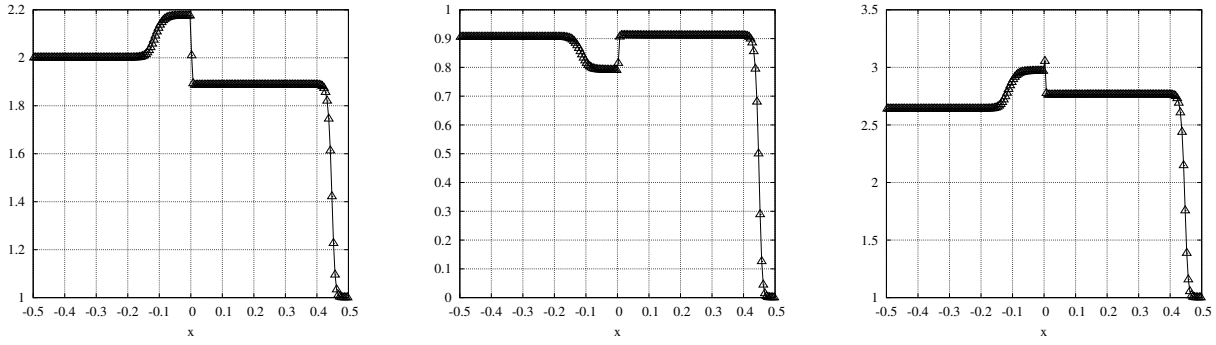


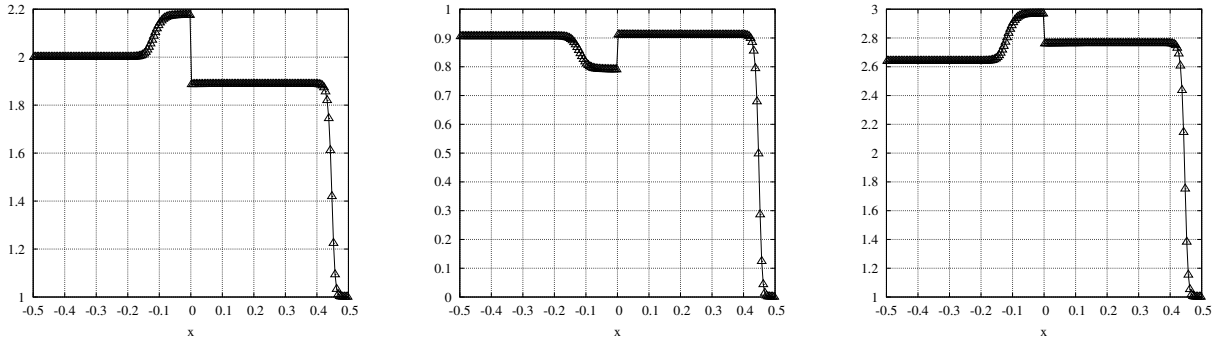
(a) Density (b) Velocity (c) Pressure

Figure 3. Conservative coupling: results for the two-Riemann problem method.



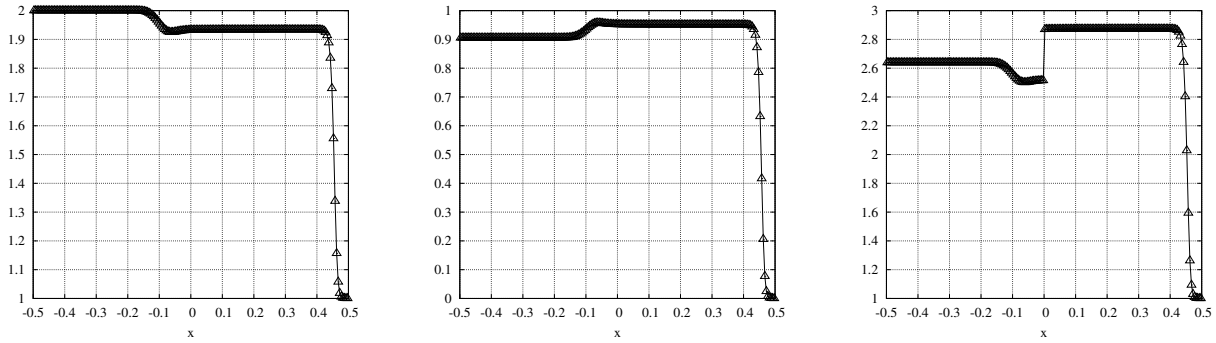
(a) Density (b) Velocity (c) Pressure

Figure 4. Conservative coupling: results for the Relaxation approach.



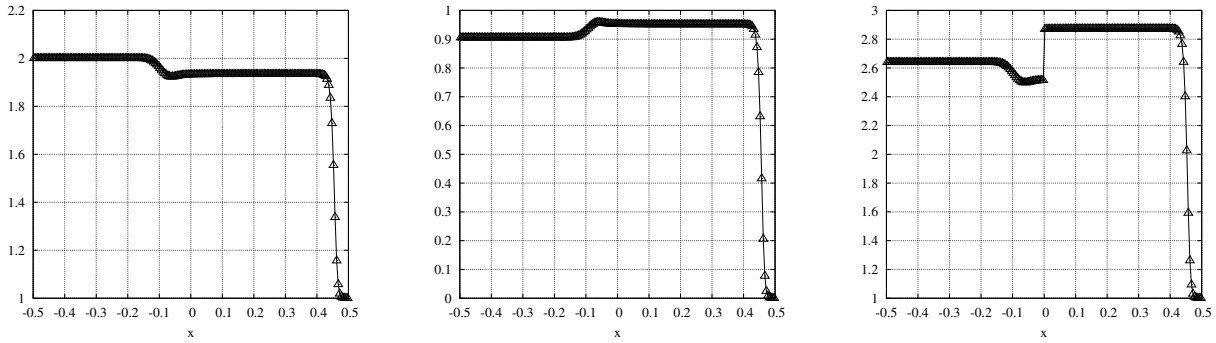
(a) Density (b) Velocity (c) Pressure

Figure 5. Conservative coupling: results for the Relaxation approach with weight $\mathcal{M}^e_{\rho\pi}$.



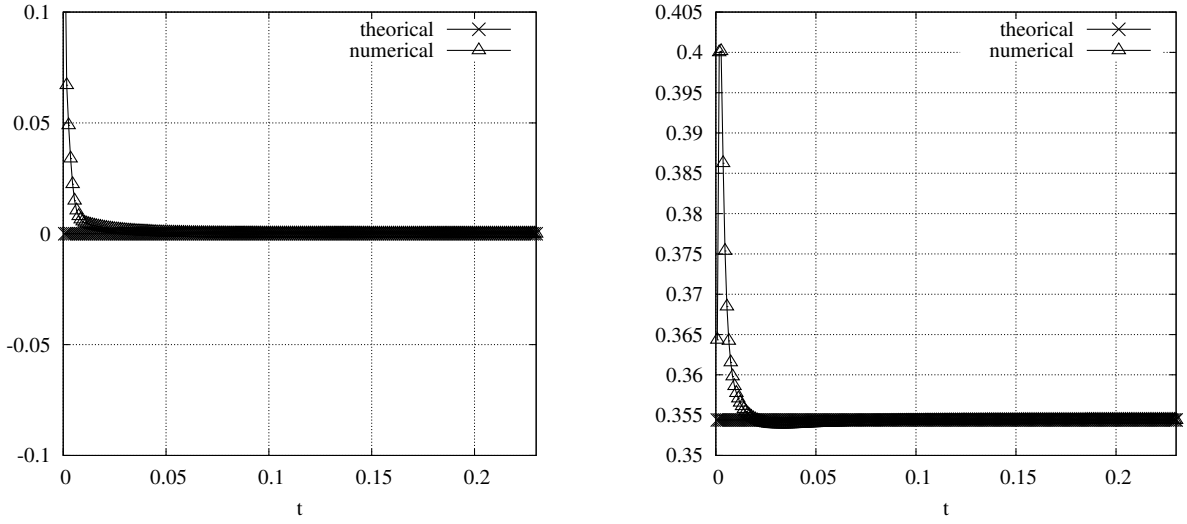
(a) Density (b) Velocity (c) Pressure

Figure 6. Non conservative coupling: results for the two-Riemann problem method.



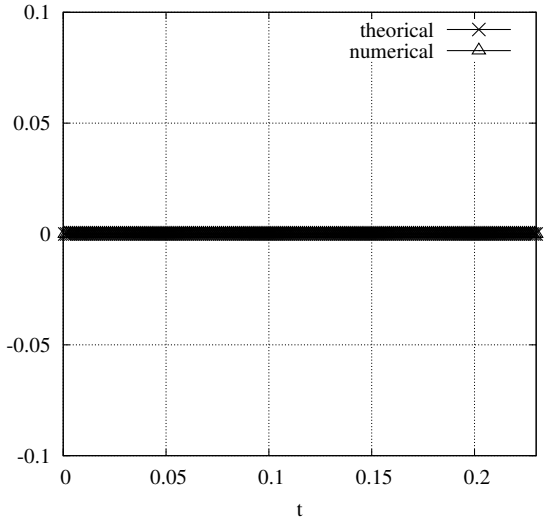
(a) Density (b) Velocity (c) Pressure

Figure 7. Non conservative coupling: results for the Relaxation approach.

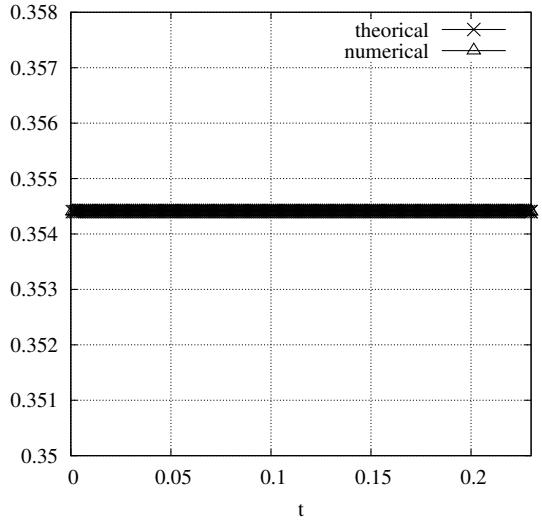


(a) Two Riemann problems: conservative coupling (b) Two Riemann problems: non conservative coupling

Figure 8. Comparison between the numerical value of the impulsion flux jump at the interface and the real weight value for the two-Riemann problem approach.



(a) Relaxation approach: conservative coupling



(b) Relaxation approach: non conservative coupling

Figure 9. Comparison between the numerical value of the impulsion flux jump at the interface and the real weight value for the relaxation approach.

References

[1]Ambroso, A., Chalons, C., Coquel, F., Godlewski, E., Lagoutiere, F., Raviart, Seguin, N., P.-A., and Hérard, J.-M., "Coupling of multiphase flow models," *proceedings of Nureth11*, 2005.

[2]Ambroso, A., Chalons, C., Coquel, F., Godlewski, E., Lagoutiere, F., Raviart, P.-A., and Seguin, N., "The coupling of homogeneous models for two-phase flows," *Int. J. Finite Volumes*, Vol. 4(1), 2007, pp. 1–39.

[3]Ambroso, A., Chalons, C., Coquel, F., Godlewski, E., Lagoutiere, F., Raviart, P.-A., and Seguin, N., "Extension of interface coupling to general coupling Lagrangian systems," *Accepted for publication in Math. Com.*, 2007.

[4]Ambroso, A., Hérard, J.-M. and Hurisse, O., "A method to couple HEM and HRM two-phase flow models," *Submitted*, 2006.

[5]Chalons, C., *Bilans d'entropie discrets dans l'approximation numérique des chocs non classiques. Application aux équations de Navier-Stokes multi-pression 2D et à quelques systèmes visco-capillaires*, Ph.D. thesis, Ecole polytechnique, France, 2002.

[6]Diehl, S., "On scalar conservation laws with point source and discontinuous flux function," *SIAM J. Numer. Anal*, Vol. 26, 1995, pp. 1425–1451.

[7]Galié, T., *Couplage interfacial de modèles pour la thermoohydraulique des réacteurs*, Ph.D. thesis, In preparation.

[8]Godlewski, E., Le Thanh, K.-C., and Raviart, P.-A., "The numerical interface coupling of nonlinear hyperbolic systems of conservation laws. II. The case of systems," *M2AN Math. Model. Numer. Anal.*, Vol. 39(4), 2005, pp. 649–692.

[9]Godlewski, E. and Raviart, P.-A., "The numerical interface coupling of nonlinear hyperbolic systems of conservation laws. I. The scalar case," *Numer. Math.*, Vol. 97(1), 2004, pp. 81–130.

[10]Greenberg, J. M., Roux, A. Y. L., Baraille, R., and Noussair, A., "Analysis and approximation of conservation laws with source terms," *SIAM J. Numer. Anal.*, Vol. 34, 1997, pp. 1980–2007.

[11]Hérard, J.-M., "Schemes to couple flows between free and porous medium," *proceedings of AIAA*, 2005, pp. 2005–4861.

[12]Hérard, J.-M. and Hurisse, O., "Coupling two and one-dimensional models through a thin interface," *proceedings of AIAA*, 2005, pp. 2005–4718.

[13]Hérard, J.-M. and Hurisse, O., "Boundary conditions for the coupling of two-phase flow models," *Submitted for contribution to the* 18^{th} *AIAA CFD conference*.

[14]Jin, S. and Xin, Z., "The relaxation schemes for systems of conservation laws in arbitrary space dimensions," *Comm. Pure Appl. Math.*, Vol. 48, 1995, pp. 235–276.