Mathematical Models and Methods in Applied Sciences

GODUNOV-TYPE SCHEMES FOR HYPERBOLIC SYSTEMS WITH PARAMETER DEPENDENT SOURCE. THE CASE OF EULER SYSTEM WITH FRICTION

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Well balanced or asymptotic preserving schemes are receiving an increasing amount of interest. This paper gives a precise setting for studying both properties in the case of Euler system with friction. We derive a simple solver which, by construction, preserves discrete equilibria and reproduces at the discrete level the same asymptotic behavior as that of the solutions of the continuous system. Numerical illustrations are convincing and show that not all methods share these properties.

Keywords: hyperbolic systems of conservation laws; source term; equilibrium; asymptotic behavior; Riemann solver; consistency; relaxation scheme.

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1. Introduction

We are interested in deriving schemes having some 'well-balanced' and 'asymptotic preserving' properties for the approximation of a nonlinear hyperbolic system with source term

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

$\mathbf{U}(x,t) \in \Omega \subset \mathbb{R}^n$. First, we say that a scheme is *well-balanced* if it preserves, in some sense which has to be precised, *stationary* solutions of (1.1) which by definition satisfy

$$\frac{d}{dx}\mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}). \tag{1.2}$$

Now, assuming that in (1.1), the source $\mathbf{S}(\mathbf{U}) = \mathbf{S}(\mathbf{U};\alpha)$ depends on some scalar parameter α in such a way that the solution $\mathbf{U} = \mathbf{U}_\alpha$ depends smoothly on α, the solutions of system (1.1) may have some typical asymptotic long-time behavior as $\alpha \to \infty$, and we want the scheme to preserve in some sense this behavior. The property will then be refered to as *asymptotic preserving*. We will construct below approximate Riemann solvers leading to Godunov-type schemes having this property when applied to the model problem of gas dynamics equations with gravity and friction, and α will be a friction parameter.

The problem of deriving schemes that work uniformly well with respect to a parameter which can become large has been often addressed to (in particular for the transition from kinetic to hydrodynamic equations, see Ref. 8 for example). Let us emphasize that the question is not only linked to the presence of stiff source terms. Indeed, a scaling wrt. the time variable will also be introduced, so that the limit behavior is governed by a reduced system, as we will see in section 2.2 below. Such problems have already been analyzed in other contexts (for the transport equation in the diffusive limit in Ref. 28 for instance, see also Ref. 31). Before we focus on our model problem, we mention several recent works related to the subject of preserving equilibrium and asymptotic properties in the case of a diffusive limit system: first we mention the pioneering papers Ref. 25 and Ref. 27, which underline the importance of the asymptotics and analyze the problem in the semi-discrete setting. More recently, Ref. 5 considers the same model problem as we do, but in the barotropic case and without gravity. We also mention some contributions for deriving asymptotic preserving and well balanced schemes for other model equations such as Ref. 26 for multiscale kinetic equations, Ref. 18, 19 for the one-dimensional Goldstein-Taylor system, for radiative transfer models: Ref. 20, 7 6, 3 and we mention the recent paper Ref. 12 where more references can be found.

The outline of the paper is as follows. In the next section, we give a precise example of what we mean by 'asymptotic behavior' on the problem of gas dynamics equations with gravity and friction; we present the model in both the Eulerian and Lagangian frames and begin by computing the possible stationary solutions before we study the asymptotic behavior. In section 3, we recall the concept of simple

approximate Riemann solver for (1.1) introduced in Ref. 13, 15, we precise the notions of discrete equilibrium and well-balanced numerical scheme in this setting, then we apply the methodology to the model example of the gas dynamics equations with friction and gravity terms in Lagrangian coordinates. In section 4, we derive the scheme for the Euler system and we assess the well-balanced and asymptotic preserving properties of the resulting scheme. In section 5, following Ref. 9 and Ref. 10, 11, we introduce a larger relaxation system with a so called *potential* and derive a relaxation scheme for the Euler system which is shown to coincide with the previous one, and this other point of view enlights some nice features of the scheme. At last we provide some numerical illustrations in section 6.

Let us now focus on the problem of gas dynamics equations with gravity and friction. It is an illustrative example and for that reason, we present the model with some details.

2. Model problem: gas dynamics equations with gravity and friction

Let us consider the gas dynamics equations with gravity and friction. In Eulerian coordinates, the system writes

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) = \varrho(gu - \alpha\psi(u)), \end{cases} \tag{2.1}$$

where $\varphi(u)$ and $\psi(u)$ model friction terms and $\alpha > 0$ is some constant coefficient which can become very large; g is a gravity constant. The functions $\varphi(u)$ and $\psi(u)$ satisfy $\varphi(0) = \psi(0) = 0, \psi'(0) = 0$, φ increasing. In what follows, we will mainly consider the commonly used friction terms

$$\begin{cases} \varphi(u) = |u|^\chi u, \\ \psi(u) = a|u|^{\chi+2}, \quad 0 \le a \le 1, \end{cases} \tag{2.2}$$

with $\chi \ge 0$, $\chi = 0$ for a linear friction or $\chi = 1$ for a quadratic friction term and a is some constant. A frequently encountered case when studying compressible flow is $a = 1$, then $\psi(u) = \varphi(u)u$. We will also consider the case $\psi(u) = a\varphi(u)u$ with $a < 1$. The energy e satisfies $e = \varepsilon + u^2/2$, ε is the internal energy and the pressure law $p = p(\varrho, \varepsilon)$ is a given function satisfying some usual assumptions; p will also be expressed either in terms of (τ, ε), $p = \check{p}(\tau, \varepsilon)$ (with $\tau = \frac{1}{\varrho}$ the specific volume), or in terms of $(\tau, \eta), \eta$ the entropy, $p = \tilde{p}(\tau, \eta)$.

It will be interesting, for the derivation of our numerical scheme, to write the system (2.1) in Lagrangian coordinates, because the computations are known to be

much easier. Denoting by m the mass variable, the Lagrangian formulation writes

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m p = g - \alpha\varphi(u), \\ \partial_t e + \partial_m (pu) = gu - \alpha\psi(u). \end{cases} \tag{2.3}$$

Recall that, denoting by η the mathematical specific entropy (-η is the physical specific entropy) which satisfies

$$-T d\eta = d\varepsilon + p d\tau$$

where T is the temperature, we obtain for a smooth solution of (2.1) the equation

$$\partial_t(\varrho\eta) + \partial_x(\varrho\eta u) = -\frac{\alpha}{T}\varrho(u\varphi(u) - \psi(u)), \tag{2.4}$$

while for a smooth solution of (2.3), we get

$$\partial_t \eta = -\frac{\alpha}{T}\alpha(u\varphi(u) - \psi(u)) \tag{2.5}$$

and for (2.2), $-T\partial_t\eta = (1-a)\alpha|u|^{\chi+2}$. In particular, for $a = 1$, the friction involves no entropy dissipation whereas for $a = 0$ (i.e., for $\psi(u) = 0$), the whole friction is transformed into internal energy.

2.1. *Stationary solutions*

Let us consider stationary solutions of system (2.3). They are characterized by

$$\begin{cases} \frac{du}{dm} = 0, \\ \frac{dp}{dm} = g - \alpha\varphi(u), \\ \frac{d}{dm}(pu) = gu - \alpha\psi(u) \end{cases} \tag{2.6}$$

hence the velocity u is constant: $u = \overline{u}$. In the lines below, we will also note the quantities which should be constant by overlining them. On the one hand, the equations

$$\begin{cases} u = 0, \\ \frac{dp}{dm} = g \end{cases} \tag{2.7}$$

provide equilibrium stationary solutions. On the other hand, if $\overline{u} \neq 0$, we must have

$$\psi(u) = u\varphi(u),$$

for all u. Thus, if $a \neq 1$, there is no stationary solution with a non zero velocity. If $a = 1$, $\frac{dp}{dm}$ is again constant and solving the system

$$\begin{cases} u = \overline{u}, \\ \frac{dp}{dm} = g - \alpha\varphi(\overline{u}) \end{cases} \tag{2.8}$$

provides all possible stationary solutions of (2.3) (note that formula (2.8) holds in all cases, but if $\overline{u} \neq 0$, it supposes $a = 1$).

For Euler system (2.1), stationary solutions are characterized by

$$\begin{cases} \frac{d}{dx}(\varrho u) = 0, \\ \\ \frac{d}{dx}(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \\ \frac{d}{dx}((\varrho e + p)u) = \varrho(gu - \alpha\psi(u)). \end{cases}$$

Hence ϱu is now constant. Again, if $u = 0$, solving

$$\begin{cases} u = 0, \\ \frac{dp}{dx} = \varrho g \end{cases} \tag{2.9}$$

provides a classical equilibrium stationary solution of (2.1) where the gravity term balances the pressure gradient. We now focus on stationary solutions satisfying moreover $u = \overline{u}$ constant. If $\overline{u} \neq 0$, we must have $\varrho = \overline{\varrho}$ constant and

$$\begin{cases} \frac{dp}{dx} = \overline{\varrho}(g - \alpha\varphi(\overline{u})), \\ \\ \overline{\varrho u}\frac{d\varepsilon}{dx} + \overline{u}\frac{dp}{dx} = \overline{\varrho}(g\overline{u} - \alpha\psi(\overline{u})) \end{cases} \tag{2.10}$$

and thus

$$\begin{cases} \frac{dp}{dx} = \overline{\varrho}(g - \alpha\varphi(\overline{u})), \\ \\ \overline{u}\frac{d\varepsilon}{dx} = \alpha(\overline{u}\varphi(\overline{u}) - \psi(\overline{u})). \end{cases} \tag{2.11}$$

If $\psi(u) = \varphi(u)u$, we get $\frac{d\varepsilon}{dx} = 0$ and $\varepsilon = \overline{\varepsilon}$,

$$p = p(\overline{\varrho}, \overline{\varepsilon}) = \overline{p}.$$

Thus if $a = 1$, we obtain a constant stationary solution of (2.1) given by

$$\begin{cases} \varrho = \overline{\varrho}, \quad \varepsilon = \overline{\varepsilon}, \\ \\ \overline{u} = \varphi^{-1}(g/\alpha). \end{cases} \tag{2.12}$$

For $a < 1$,we get by (2.10) a non trivial equation for $\varepsilon = \varepsilon(x)$

$$\frac{d\varepsilon}{dx} = \alpha(\varphi(\overline{u}) - \frac{\psi(\overline{u})}{\overline{u}}).$$

Then it yields that, if $a < 1$, stationary solutions with constant $u = \overline{u} \neq 0$ are such that ε grows linearly in the domain (let us pass over the problem of boundary in silence). For instance, in the case of an ideal gas $p = (\gamma - 1)\varrho\varepsilon$, (2.11) yields

$$\frac{d\varepsilon}{dx} = \frac{1}{\gamma - 1}(g - \alpha\varphi(\overline{u})) = \alpha(\varphi(\overline{u}) - \frac{\psi(\overline{u})}{\overline{u}})$$

and $\overline{u}$ satisfies

$$g = \alpha(\gamma\varphi(\overline{u}) - (\gamma-1)\frac{\psi(\overline{u})}{\overline{u}}) = \alpha(\gamma - (\gamma-1)a)\varphi(\overline{u}).$$

The resulting stationary solution is thus given by

$$\begin{cases} \varrho = \overline{\varrho}, \\ \overline{u} = \varphi^{-1}(\frac{g}{\alpha}\ \frac{1}{1+(\gamma-1)(1-a)}) \\ \frac{d\varepsilon}{dx} = \alpha(1-a)\varphi(\overline{u}) \end{cases} \tag{2.13}$$

with

$$\varphi^{-1}(v) = |v|^{\frac{\chi}{\chi+1}}v.$$

The friction term with $\psi < u\varphi$ naturally induces an increase of internal energy.

Remark 2.1. Stationary solutions for Lagrange formulation are in correspondance with transport waves solution for Euler, i.e. $u = \overline{u}$ constant, and ϱ, ε functions of $x - \overline{u}t$. For such a transport wave solution, all convection terms of the form $D_t. = \partial_t. + \overline{u}\partial_x.$ vanish. There remains only when $a = 1$, $\partial_x p = \varrho(g - \alpha\varphi(\overline{u}))$, corresponding to (2.8). When $\overline{u} = 0$, we recover the steady solutions (2.9). We can have a constant state (2.12) with $\overline{u} \neq 0$, then $\partial_x p = 0$, and in Lagrangian formulation, it is a special case of (2.8) with $p = \overline{p}$ constant.

The above energy increasing (in space) stationary solution for Euler (see (2.13) is not stationary in a Lagrangian frame; u, ϱ are constant but the energy increases (with time) since it satisfies $\partial_t\varepsilon = \alpha(\overline{u}\varphi - \psi) = \alpha\overline{u}(1-a)\varphi(\overline{u})$. □

2.2. *Formal study of the asymptotic behavior*

Let us consider first the asymptotic behavior of the solutions of system (2.3) or of system (2.1) as the friction parameter $\alpha \to +\infty$. Let us start with (2.3). Assume that we can write some asymptotic expansions in powers of $\frac{1}{\alpha}$:

$$\begin{cases} \tau = \tau^0 + \frac{1}{\alpha}\tau^1 + \dots, \\ u = u^0 + \frac{1}{\alpha}u^1 + \dots, \\ \varepsilon = \varepsilon^0 + \frac{1}{\alpha}\varepsilon^1 + \dots \end{cases} \tag{2.14}$$

Together with the pressure law $p = \tilde{p}(\tau, \varepsilon)$, one finds

$$p = p^0 + \frac{1}{\alpha}p^1 + \dots, \quad p^0 = p(\tau^0, \varepsilon^0)$$

whereas

$$e = \varepsilon + \frac{1}{2}u^2 = e^0 + \frac{1}{\alpha}e^1 + \dots, \quad e^0 = \varepsilon^0 + \frac{1}{2}(u^0)^2.$$

Substituting in (2.3) the expansions for τ, u and ε, we get

$$\begin{cases} \partial_t \tau^0 - \partial_m u^0 + \frac{1}{\alpha}(\partial_t \tau^1 - \partial_m u^1) + \cdots = 0, \\ \partial_t u^0 + \partial_m p^0 + \frac{1}{\alpha}(\partial_t u^1 + \partial_m p^1) + \cdots = -\alpha\varphi(u^0) + g - \varphi'(u^0)u^1 + \ldots, \\ \partial_t e^0 + \partial_m (p^0 u^0) + \frac{1}{\alpha}(\partial_t e^1 + \partial_m (pu)^1) + \cdots = -\alpha\psi(u^0) + gu^0 - \psi'(u^0)u^1 + \ldots \end{cases}$$

Looking at the terms of order $\mathcal{O}(\alpha)$, we get

$$\varphi(u^0) = 0, \ \psi(u^0) = 0,$$

which yields $u^0 = 0$, whereas zeroth order terms (with order $\mathcal{O}(1)$) write

$$\begin{cases} \partial_t \tau^0 - \partial_m u^0 = 0, \\ \partial_t u^0 + \partial_m p^0 = g - \varphi'(u^0)u^1, \\ \partial_t e^0 + \partial_m (p^0 u^0) = gu^0 - \psi'(u^0)u^1 \end{cases}$$

and since $u^0 = 0$, together with $\psi'(0) = 0$, we get

$$\begin{cases} \tau^0 = \tau^0(m), \\ \partial_m p^0 = g - \varphi'(0)u^1, \\ \varepsilon^0 = \varepsilon^0(m). \end{cases}$$

Thus $p^0 \equiv \check{p}(\tau^0(m), \varepsilon^0(m))$ depends only on m and when $\chi = 0$ (linear friction), $\varphi'(0) = 1$, thus in this case we obtain

$$\begin{cases} \tau^0 = \tau^0(m), \quad \varepsilon^0 = \varepsilon^0(m), \quad p = p^0(m) \\ \frac{dp^0}{dm} = g - u^1. \end{cases}$$

When $\chi > 0$, $\varphi'(0) = 0$ and we get

$$\begin{cases} \tau^0 = \tau^0(m), \quad \varepsilon^0 = \varepsilon^0(m), \\ \frac{dp^0}{dm} = g, \end{cases}$$

which implies that the initial data should be 'well prepared' (if not, we have a boundary layer in time). To conclude this *formal* argument, it is natural to assume that

$$u \to u^0 = 0 \quad \text{as } \alpha \to +\infty. \tag{2.15}$$

Let us now analyze the *long time behavior* of the solutions of (2.3) for large friction. We first perform a change of variables: setting $\beta = \alpha^{\frac{1}{\chi+1}}$, we define

$$t = \beta s, \quad v_\beta(m,s) = \beta u(m,t), \tau_\beta(m,s) = \tau(m,t), \varepsilon_\beta(m,s) = \varepsilon(m,t). \tag{2.16}$$

With this scaling, we can indeed study the long time behavior of the solution of (2.3) for high friction since we note that the scaling for the time variable and the velocity are coherent with (2.15) if we let $\alpha \to \infty$. Using (2.2), system (2.3) now writes (dropping the subscript β)

$$\begin{cases} \partial_s \tau - \partial_m v = 0, \\ \frac{1}{\beta^2}\partial_s v + \partial_m p = g - \varphi(v), \\ \partial_s \varepsilon + \frac{1}{2\beta^2}\partial_s v^2 + \partial_m (pv) = gv - \psi(v), \end{cases} \tag{2.17}$$

together with $p = \tilde{p}(\tau, \varepsilon)$. The terms of order 0 wrt. $\frac{1}{\beta}$, should vanish, leading to

$$\begin{cases} \partial_s \tau - \partial_m v = 0, \\ \partial_m p = g - \varphi(v), \\ \partial_s \varepsilon + \partial_m (pv) = gv - \psi(v). \end{cases} \tag{2.18}$$

Thus (τ, ε) satisfies the following system

$$\begin{cases} \partial_s \tau - \partial_m v = 0, \\ \partial_s \varepsilon + \partial_m (pv) = gv - \psi(v), \end{cases} \tag{2.19}$$

where $v = v(\tau, \varepsilon)$ is given by the second equation in (2.18), which for a linear friction ($\chi = 0$) writes

$$v = g - \partial_m p, \text{with } p = \check{p}(\tau, \varepsilon),$$

and system (2.19) in that case becomes

$$\begin{cases} \partial_s \tau + \partial^2_{mm} p = 0, \\ \partial_s \varepsilon + 2(1-a) g \partial_m p + a(\partial_m p)^2 - \partial_m (p \partial_m p) = (1-a) g^2. \end{cases}$$

Now if moreover $a = 1$, we get

$$\begin{cases} \partial_s \tau + \partial^2_{mm} p = 0, \\ \partial_s \varepsilon - p \partial^2_{mm} p = 0 \\ v = g - \partial_m p, \ p = \check{p}(\tau, \varepsilon). \end{cases} \tag{2.20}$$

Let us consider now the asymptotic behavior of the solutions of the system writen in Eulerian coordinates with a similar scaling $t = \beta s$, $v = \beta u$ as in (2.16). Then (2.1) becomes

$$\begin{cases} \partial_s \varrho + \partial_x (\varrho v) = 0, \\ \frac{1}{\beta^2}(\partial_s (\varrho v) + \partial_x (\varrho v^2)) + \partial_x p = \varrho (g - \varphi(v)), \\ \frac{1}{2\beta^2}(\partial_s (\varrho v^2) + \partial_x (\varrho v^3)) + \partial_s (\varrho \varepsilon) + \partial_x ((\varrho \varepsilon + p) v) = \varrho (gv - \psi(v)). \end{cases} \tag{2.21}$$

Again, the zeroth order terms wrt. $\frac{1}{\beta}$ should vanish

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \partial_x p = \varrho(g - \varphi(v)), \\ \partial_s(\varrho\varepsilon) + \partial_x((\varrho\varepsilon + p)v) = \varrho(gv - \psi(v)) \end{cases}$$

and (ϱ, ε) satisfies

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \partial_s(\varrho\varepsilon) + \partial_x((\varrho\varepsilon + p)v) = \varrho(gv - \psi(v)) \\ v = \varphi^{-1}(g - \frac{1}{\varrho}\partial_x p), \quad p = p(\varrho, \varepsilon). \end{cases} \tag{2.22}$$

If $a = 1$, i.e. $\psi = u\varphi$, the energy equation can be simplified and we get

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \partial_s(\varrho\varepsilon) + \partial_x(\varrho\varepsilon v) + p\partial_x v = 0, \\ v = \varphi^{-1}(g - \frac{1}{\varrho}\partial_x p), \quad p = p(\varrho, \varepsilon), \end{cases} \tag{2.23}$$

and assuming moreover a linear friction, the energy equation writes

$$\partial_s(\varrho\varepsilon) + \partial_x(\varrho\varepsilon v) - p\partial_x(\frac{1}{\varrho}\partial_x p) = 0.$$

in this case (linear friction and $a = 1$), (2.23) becomes, with $p = p(\varrho, \varepsilon)$,

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \partial_s(\varrho\varepsilon) + \partial_x(\varrho\varepsilon v) - p\partial_x(\frac{1}{\varrho}\partial_x p) = 0, \\ v = g - \frac{1}{\varrho}\partial_x p, \ p = p(\varrho, \varepsilon). \end{cases} \tag{2.24}$$

This is the exact analog of (2.20). Indeed, by the change of frame from Eulerian to Lagrangian coordinates, the first equation in (2.24) gives (after the same scaling) $\partial_s \tau - \partial^2_{mm} p = 0$, idem for the second equation.

Remark 2.2. In the barotropic case, Euler system writes

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho v) = 0, \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \quad p = p(\varrho), \end{cases} \tag{2.25}$$

and we get (after scaling) a nonlinear parabolic equation for the zeroth order terms in ϱ

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \\ v = \varphi^{-1}(g - \frac{1}{\varrho}\partial_x p), \end{cases} \tag{2.26}$$

with now $p = p(\varrho)$. In particular for a linear friction, v is given by

$$v = g - \frac{1}{\varrho}\partial_x p$$

we find a Darcy-type model

$$\frac{\partial}{\partial s}\varrho + g\frac{\partial}{\partial x}\varrho - \frac{\partial^2}{\partial x^2}p(\varrho) = 0. \tag{2.27}$$

In the Lagrangian frame, the analogous of (2.27) (for linear friction) writes

$$\begin{cases} \partial_s \tau + \partial^2_{mm}(p) = 0, \\ \\ v = g - \partial_m p, \end{cases} \tag{2.28}$$

with now $p = \tilde{p}(\tau)$. In this barotropic case, there are several existing theoretical results[30,23,24] justifying the formal analysis above. □

There would remain to study (2.22) and to extend the theoretical results relative to the barotropic case to the full system with energy. Thus two natural theorical questions arise: is (2.22) well-posed? and, shortly speaking, does (2.21) converge to this generalized Darcy model (2.22) as $\alpha \to +\infty$? with similar questions for the Lagrangian case. Answering these questions is beyond the scope of the present work which is rather devoted to the numerical approximation of the problem. We will construct a (consistent, stable) scheme for system (2.1) which in the limit $\alpha \to \infty$, and after the same scaling (2.16), gives a (consistent, stable) scheme for the limit system (2.22), this is what is usually called asymptotic preserving property (AP). In order to do that, we use the simpler Lagrangian frame (2.3) and (2.20) and come back to the Euler setting following Ref. 15. Moreover, concerning the asymptotic property, we emphasize that the scaling we have performed is compatible with the change of frame Euler $\leftrightarrow$ Lagrange in that the total derivative is relevant at both scales, since $\partial_t + u\partial_x$ becomes after scaling $\partial_s + v\partial_m$.

Besides the references already given, we mention some recent work precisely related to the subject. In Ref. 5, the authors consider the barotropic case (2.25) without gravity: the approach uses a classical finite volume scheme together with the upwinding of source terms involving the reconstruction of interface variables while preserving Darcy steady states i.e. solutions of (2.27) (where g is set to zero and thus $\partial_x p = -\varrho v$). It supposes, in order to prove the AP property, a restrictive hypothesis for the basis scheme which is not valid for all schemes.

We emphasize that our approach using simple approximate Riemann solvers and Godunov-type schemes[22] is very straightforward, and we begin by presenting this approach because of its relative simplicity.

However, it is interesting to see that the resulting scheme can be obtained from a quite different approach, using first ideas introduced in Ref. 9 for the Euler equation with gravity in the case without friction ($\alpha = 0$), together with relaxation schemes (note that the relation between approximate Riemann solvers and relaxation schemes is well established in Ref. 29, see Ref. 4). The authors in Ref. 9 derive a well-balanced scheme for preserving stationary solutions with zero velocity $\partial_x p = -\varrho g$ (satisfying (2.9)). In the same spirit, we introduce a so called *potential*, and derive a relaxation scheme as developed in Ref. 11, for a larger system which is in conservative form and the scheme now relies on an exact Riemann solver for this enlarged system. With this point of view of relaxation schemes, we can understand how to get the desired properties (entropy, well-balanced, AP), at least in an heuristic way, since we can use the corresponding properties well-known at the continuous (PDE) level. Moreover, the 'equivalence' between Lagrangian and Eulerian frame also makes it clear that if the AP property holds in one frame, it holds in the other, provided the velocities are given consistent definitions.

3. Numerical methodology

The aim of this section is to propose a fairly general strategy to derive a consistent and asymptotic preserving scheme for the nonlinear system with source term (1.1). This system is assumed to be hyperbolic and we denote by $\lambda_k(\mathbf{U})$, $1 \leq k \leq n$, the n real eigenvalues of the Jacobian matrix $\mathbf{A}(\mathbf{U}) = \mathbf{F}'(\mathbf{U})$. We only consider entropy solutions of (1.1) that is solutions satisfying the entropy inequality

$$\partial_t \eta(\mathbf{U}) + \partial_x \mathcal{Q}(\mathbf{U}) \leq \sigma(\mathbf{U}) \tag{3.1}$$

with $\sigma(\mathbf{U}) = \eta'(\mathbf{U}) \cdot \mathbf{S}(\mathbf{U})$. As is customary, the entropy entropy-flux pair $(\eta, \mathcal{Q})$ is such that $\mathbf{U} \mapsto \eta(\mathbf{U})$ is convex and $\mathcal{Q}'(\mathbf{U}) = \eta'(\mathbf{U}) \cdot \mathbf{F}'(\mathbf{U})$.
We first review the concept of simple approximate Riemann solver for (1.1) developed in Ref. 13, 14, 15 following Ref. 22, see also, for a similar approach, the wave propagation method,[2] and the idea of well-balanced numerical scheme (notion introduced in Ref. 9, 21 and 17). Then, we apply the methodology to the first example of the gas dynamics equations with friction and gravity terms in Lagrangian coordinates (2.3). At last, we assess the well-balanced and asymptotic preserving properties of the resulting scheme.

3.1. *Simple approximate Riemann solvers*

Solving the Riemann problem amounts to find the solution of (1.1), (3.1) with the following piecewise constant initial data

$$\mathbf{U}(x,0) = \mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L, & x < 0, \\ \mathbf{U}_R, & x > 0, \end{cases} \tag{3.2}$$

for any given $\mathbf{U}_L$ and $\mathbf{U}_R$ in Ω. Unlike the homogeneous case ($\mathbf{S}(\mathbf{U}) = 0$), the exact solution of (1.1)-(3.1)-(3.2) that we denote $\mathbf{W}(x,t;\mathbf{U}_L,\mathbf{U}_R)$ is no longer self-similar, as it also occurs for generalized Riemann problem (see Ref. 16). Notwithstanding, notice that an approximate Riemann solver $\widetilde{\mathbf{W}}(x,t;\mathbf{U}_L,\mathbf{U}_R)$ associated with $\mathbf{W}(x,t;\mathbf{U}_L,\mathbf{U}_R)$ may be self-similar as in the homogeneous case $\mathbf{S}(\mathbf{U}) = 0$. In this paper, we will focus on simple approximate Riemann solvers of the following form

$$\widetilde{\mathbf{W}}(\frac{x}{t};\mathbf{U}_L,\mathbf{U}_R) = \begin{cases} \mathbf{U}_1 = \mathbf{U}_L, & \frac{x}{t} < \lambda_1, \\ \vdots & \\ \mathbf{U}_k, & \lambda_{k-1} < \frac{x}{t} < \lambda_k, \quad k = 2,..,l, \\ \vdots & \\ \mathbf{U}_{l+1} = \mathbf{U}_R, & \frac{x}{t} > \lambda_l, \end{cases} \tag{3.3}$$

with $\mathbf{U}_k$ and $\lambda_k = \lambda_k(\mathbf{U}_L,\mathbf{U}_R)$, $k = 1,..,l$, $l \leq n$, to be defined (note that we have also noted by λ_k the different wave velocities in (3.3) since there will be no ambiguity with the eigenvalues of the Jacobian). The general setting is treated in Ref. 15. This approximate Riemann solver is self-similar, then from now on and with a little abuse in the notations, we identify $\widetilde{\mathbf{W}}(x,t;\mathbf{U}_L,\mathbf{U}_R)$ and $\widetilde{\mathbf{W}}(\frac{x}{t};\mathbf{U}_L,\mathbf{U}_R)$. The influence of the source term $\mathbf{S}(\mathbf{U})$ will appear in the definition of the intermediate states $\mathbf{U}_k$, $k = 2,..,l$.

Remark 3.1. Note that in (3.3), the intermediate states and velocities will also depend on the grid size Δx (and might depend on the time step Δt) because the solver 'incorporates' the influence of the source term. Indeed one can interpret the formula for the approximate Riemann solver by first solving an 'exact' Riemann problem where the source term has been discretized as a Dirac measure (following,[2] where the source term is 'replaced by a sum of Delta-function sources' at cell edges) with weight the integral of $\mathbf{S}$ on a cell: $\int_{-\Delta x/2}^{\Delta x/2} \mathbf{S}(\mathbf{U})dx$. Then the solution of this problem is indeed self-similar but with intermediate states and velocities depending on Δx, hence it must be the same for the approximate solver. In the exemple of Euler system below, this dependence appears explicitly through the choice of $\tilde{u} = u^*$ satisfying (3.31). □

Following Gallice,[14,13] a suitable approximate Riemann solver for (1.1) is subject to the following consistency property.

Definition 3.1. A simple approximate Riemann solver (3.3) is said to be consistent with the integral form of (1.1) if there exists a function $\widetilde{\mathbf{S}}(\xi,\tau;\mathbf{U}_L,\mathbf{U}_R)$ such that for $\Delta t, \Delta x$ satisfying

$$\max_{1\leq k\leq l} \mid \lambda_k(\mathbf{U}_L,\mathbf{U}_R) \mid \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \tag{3.4}$$

we have

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) - \Delta x \widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \sum_{k=1}^{l} \lambda_k (\mathbf{U}_{k+1} - \mathbf{U}_k) \tag{3.5}$$

with

$$\lim_{\substack{\Delta x, \Delta t \to 0 \\ \mathbf{U}_L, \mathbf{U}_R \to \mathbf{U}}} \widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \mathbf{S}(\mathbf{U}). \tag{3.6}$$

Similarly, we introduce the notion of consistency with the entropy inequality (3.1).

Definition 3.2. A simple approximate Riemann solver (3.3) is said to be consistent with the integral form of (3.1) if there exists a function $\tilde{\sigma}(\xi, \tau; \mathbf{U}_L, \mathbf{U}_R)$ such that for $\Delta t/\Delta x$ satisfying (3.4) we have

$$q(\mathbf{U}_R) - q(\mathbf{U}_L) - \Delta x \, \tilde{\sigma}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) \leq \sum_{k=1}^{l} \lambda_k (\eta(\mathbf{U}_{k+1}) - \eta(\mathbf{U}_k)) \tag{3.7}$$

with

$$\lim_{\substack{\Delta x, \Delta t \to 0 \\ \mathbf{U}_L, \mathbf{U}_R \to \mathbf{U}}} \tilde{\sigma}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \sigma(\mathbf{U}). \tag{3.8}$$

Let us now introduce the points (x_j, t_n) with $x_j = j\Delta x$, $t_n = n\Delta t$, the interfaces $x_{j+1/2} = (x_j + x_{j+1})/2$ and $\mathbf{U}_j^n$ the numerical approximation of $\mathbf{U}(x, t^n)$ in the cell $C_j =]x_{j-1/2}, x_{j+1/2}[$. Given a simple approximate Riemann solver $\widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}(\frac{x}{t}; \mathbf{U}_L, \mathbf{U}_R)$, we consider the two averages

$$\begin{cases} \widetilde{\mathbf{U}}^{-}(\mathbf{U}_L, \mathbf{U}_R) = \dfrac{2}{\Delta x} \displaystyle\int_{-\frac{\Delta x}{2}}^{0} \widetilde{\mathbf{W}}(\frac{x}{\Delta t}; \mathbf{U}_L, \mathbf{U}_R) dx, \\ \\ \widetilde{\mathbf{U}}^{+}(\mathbf{U}_L, \mathbf{U}_R) = \dfrac{2}{\Delta x} \displaystyle\int_{0}^{\frac{\Delta x}{2}} \widetilde{\mathbf{W}}(\frac{x}{\Delta t}; \mathbf{U}_L, \mathbf{U}_R) dx \end{cases} \tag{3.9}$$

and define the following update formula

$$\mathbf{U}_j^{n+1} = \frac{1}{2}(\mathbf{U}_{j-\frac{1}{2}}^{n,+} + \mathbf{U}_{j+\frac{1}{2}}^{n,-}) \tag{3.10}$$

with $\mathbf{U}_{j+\frac{1}{2}}^{n,\pm} = \widetilde{\mathbf{U}}^{\pm}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ and under the usual CFL condition

$$\max_{1 \leq k \leq l} | \lambda_k(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) | \frac{\Delta t}{\Delta x} \leq \frac{1}{2}$$

for all $j \in \mathbb{Z}$. This approach is very classical in the context of approximate Riemann solvers (see Ref. 22, 16). Hereafter, such a numerical scheme is said to be *Godunov-type* if the consistency property (3.5)-(3.6) is satisfied and *entropy* satisfying if it

obeys moreover (3.7)-(3.8).
Introducing the notation

$$\widetilde{\mathbf{G}}(\mathbf{U}_L,\mathbf{U}_R)=\frac{1}{2}\left\{\mathbf{F}(\mathbf{U}_L)+\mathbf{F}(\mathbf{U}_R)-\sum_{k=1}^{l}|\lambda_k|(\mathbf{U}_{k+1}-\mathbf{U}_k)\right\}, \tag{3.11}$$

we state (see Ref. 13 for the proof):

Proposition 3.1. *Let $\widetilde{\boldsymbol{W}}(\frac{x}{t};\boldsymbol{U}_L,\boldsymbol{U}_R)$ be a simple approximate Riemann solver consistent with the integral form of (1.1) in the sense of Definition 3.1 above. Then, the numerical scheme defined by (3.10) can be written in the following conservative form:*

$$\boldsymbol{U}_j^{n+1}=\boldsymbol{U}_j^n-\frac{\Delta t}{\Delta x}(\boldsymbol{G}_{j+\frac{1}{2}}^n-\boldsymbol{G}_{j-\frac{1}{2}}^n)+\frac{\Delta t}{2}(\boldsymbol{S}_{j-\frac{1}{2}}^n+\boldsymbol{S}_{j+\frac{1}{2}}^n) \tag{3.12}$$

with $\boldsymbol{G}_{j+\frac{1}{2}}^n=\widetilde{\boldsymbol{G}}(\boldsymbol{U}_j^n,\boldsymbol{U}_{j+1}^n)$ and $\boldsymbol{S}_{j+\frac{1}{2}}^n=\widetilde{\boldsymbol{S}}(\Delta x,\Delta t;\boldsymbol{U}_j^n,\boldsymbol{U}_{j+1}^n)$ for all $j\in\mathbb{Z}$.

Formula (3.12) allows a first comment: the usual form of a conservative scheme is recovered in the absence of source term ($\mathbf{S}(\mathbf{U})=0$), while in the general setting, the influence of $\mathbf{S}(\mathbf{U})$ is taken into account by means of two interfacial contributions, namely $\mathbf{S}_{j-\frac{1}{2}}^n$ and $\mathbf{S}_{j+\frac{1}{2}}^n$.

Remark 3.2. Note however that if formula (3.12) enables us to recover the conservative formulation in the absence of source term, in the presence of these terms, the real meaningful decomposition writes in the form $\mathbf{G}_{j+\frac{1}{2}}^n-\frac{\Delta x}{2}\mathbf{S}_{j+\frac{1}{2}}^n$ and $\mathbf{G}_{j-\frac{1}{2}}^n+\frac{\Delta x}{2}\mathbf{S}_{j-\frac{1}{2}}^n$, gathering interface flux and part of source ('interface source'). Well balanced schemes are usually written with such 'non conservative' fluxes, say $\mathbf{G}_{j\mp\frac{1}{2},\pm}^n$, where the precise decomposition between flux and source in each of the two terms $\mathbf{G}_{j-\frac{1}{2},+}^n$ and $\mathbf{G}_{j+\frac{1}{2},-}^n$ may take into account some upwinding of the source as in Ref. 5 (and references therein). □

As far as the entropy inequality is concerned and introducing the notation

$$\begin{aligned}&\widetilde{\mathcal{Q}}(\mathbf{U}_L,\mathbf{U}_R)=\tfrac{1}{2}(\mathcal{Q}\big(\mathbf{U}_L)+\mathcal{Q}(\mathbf{U}_R)\big)\\&-\frac{\Delta x}{4\Delta t}\Big(\big(\eta(\mathbf{U}_R)-\eta(\widetilde{\mathbf{U}}^+(\mathbf{U}_L,\mathbf{U}_R))\big)-\big(\eta(\mathbf{U}_L)-\eta(\widetilde{\mathbf{U}}^-(\mathbf{U}_L,\mathbf{U}_R))\big)\Big),\end{aligned} \tag{3.13}$$

we have (see again Ref. 15 for the proof of this result):

Proposition 3.2. *Let $\widetilde{\boldsymbol{W}}(\frac{x}{t};\boldsymbol{U}_L,\boldsymbol{U}_R)$ be a simple approximate Riemann solver consistent with the integral form of (3.1) in the sense of Definition 3.2 above.*

Then, the numerical scheme defined by (3.10) satisfies the following discrete entropy inequality

$$\eta(\boldsymbol{U}_j^{n+1})\leq\eta(\boldsymbol{U}_j^n)-\frac{\Delta t}{\Delta x}(\mathcal{Q}_{j+\frac{1}{2}}^n-\mathcal{Q}_{j-\frac{1}{2}}^n)+\frac{\Delta t}{2}(\sigma_{j-\frac{1}{2}}^n+\sigma_{j+\frac{1}{2}}^n) \tag{3.14}$$

with $\mathcal{Q}_{j+\frac{1}{2}}^n=\widetilde{\mathcal{Q}}(\boldsymbol{U}_j^n,\boldsymbol{U}_{j+1}^n)$ and $\sigma_{j+\frac{1}{2}}^n=\widetilde{\sigma}(\Delta x,\Delta t;\boldsymbol{U}_j^n,\boldsymbol{U}_{j+1}^n)$ for all $j\in\mathbb{Z}$.

Here again, inequality (3.14) permits to recover the usual form of a discrete entropy inequality in the homogeneous case, whereas the two interfacial terms $\sigma^n_{j\pm\frac{1}{2}}$ account for the influence of the source term.

3.2. *Well-balanced numerical schemes*

This paragraph briefly reminds the so-called well-balanced property associated with a *Godunov-type* scheme following Ref. 13. The concept of well-balanced numerical scheme introduced in Ref. 21 is related to the question of preserving at the discrete level the steady solutions of (1.1). These *equilibrium* solutions satisfy by definition

$$\partial_t \mathbf{U} = 0$$

and at the continuous level, this is equivalent to

$$\partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}). \tag{3.15}$$

A relevant numerical scheme for (1.1) may be expected to preserve stationary solutions, that is discrete solutions satisfying

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n, \quad j \in \mathbb{Z}, \tag{3.16}$$

but there are many ways to obtain a discrete counterpart of (3.15); for instance, if we localize around the interface $x_{j+1/2}$, we get

$$\frac{1}{\Delta x}(\mathbf{F}(\mathbf{U}_{j+1}^n) - \mathbf{F}(\mathbf{U}_j^n)) = \mathbf{S}_{j+\frac{1}{2}}^n, \quad j \in \mathbb{Z}, \tag{3.17}$$

where $\mathbf{S}_{j+\frac{1}{2}}$ represents the $x_{j+1/2}$-interfacial contribution of the source term $\mathbf{S}$; for scheme (3.12) recall that $\mathbf{S}_{j+\frac{1}{2}}^n = \widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$.

Remark 3.3. This notion of discrete equilibrium can be defined for any finite difference type scheme, where $\mathbf{S}_{j+\frac{1}{2}}^n$ is some consistent approximation of the interfacial contribution of the source term up to first (or higher) order terms in Δx. However, there is no evidence that in the general case, without further assumption on the discrete source term $\widetilde{\mathbf{S}}$, this definition is relevant. Nothing ensures that every solution of (3.15) can be approximated by a discrete solution of (3.17). Thus, in the applications, it is important to have a proper definition of discrete equilibria, or in the above formalism, to check the corresponding properties of the discrete source term $\widetilde{\mathbf{S}}$ introduced in definition 3.1 in order that the 'physical equilibria' are indeed preserved.

Moreover, keeping in mind that we intend to respect the asymptotic behavior wrt. α, the dependence of the above mentionned $\mathcal{O}(\Delta x)$ error terms on α will have to be precisely analyzed. □

For scheme (3.14), let us introduce some stronger notions which are easier to handle. Note that by definition (3.10), a sufficient condition for (3.16) is

$$\mathbf{U}_j^n = \mathbf{U}_{j-\frac{1}{2}}^{n,+} = \mathbf{U}_{j+\frac{1}{2}}^{n,-}, \tag{3.18}$$

which motivates the following definition.

Definition 3.3. The sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ is said to be an equilibrium solution of the Godunov-type numerical scheme (3.10) if (3.18) is satisfied.

In comparison to (3.16), the main advantage of (3.18) is that this property satisfied by a stationary numerical solution can be easily connected to the discrete property (3.17) of a stationary solution of system (1.1). More precisely, the next proposition holds true.

Proposition 3.3. *An equilibrium solution of the Godunov-type numerical scheme (3.10) satisfies (3.17) for all $j \in \mathbb{Z}$.*

Proof. Let $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ be an equilibrium solution of the Godunov-type numerical scheme (3.10). We first write by definitions (3.9) and (3.10)

$$\mathbf{U}_j^{n+1} = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{\mathbf{W}}(\frac{x}{\Delta t};\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)dx - \frac{1}{2}(\mathbf{U}_{j+\frac{1}{2}}^{n,+} - \mathbf{U}_{j-\frac{1}{2}}^{n,+}). \tag{3.19}$$

But under the CFL condition (3.4) we easily show that

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{\mathbf{W}}(\frac{x}{\Delta t};\mathbf{U}_L,\mathbf{U}_R)dx = \frac{\Delta x}{2}(\mathbf{U}_L+\mathbf{U}_R) - \Delta t\sum_{k=1}^{l}\lambda_k(\mathbf{U}_{k+1}-\mathbf{U}_k),$$

so that thanks to the consistency condition (3.5)

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{\mathbf{W}}(\frac{x}{\Delta t};\mathbf{U}_L,\mathbf{U}_R)dx = \frac{\Delta x}{2}(\mathbf{U}_L+\mathbf{U}_R) - \Delta t(\mathbf{F}(\mathbf{U}_R)-\mathbf{F}(\mathbf{U}_L))$$

$$+\Delta x\Delta t\,\widetilde{\mathbf{S}}(\Delta x,\Delta t;\mathbf{U}_L,\mathbf{U}_R).$$

Plugging this equality in (3.19) gives by definition of $\widetilde{\mathbf{S}}_{j+\frac{1}{2}}^n$ in (3.12):

$$\left\{\begin{array}{l} \mathbf{U}_j^{n+1} = \dfrac{1}{2}(\mathbf{U}_j^n+\mathbf{U}_{j+1}^n) - \dfrac{\Delta t}{\Delta x}(\mathbf{F}(\mathbf{U}_{j+1}^n)-\mathbf{F}(\mathbf{U}_j^n)) \\ \qquad\qquad +\Delta t\,\mathbf{S}_{j+\frac{1}{2}}^n - \dfrac{1}{2}(\mathbf{U}_{j+\frac{1}{2}}^{n,+} - \mathbf{U}_{j-\frac{1}{2}}^{n,+}). \end{array}\right. \tag{3.20}$$

From (3.20) we infer

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{F}(\mathbf{U}_{j+1}^n)-\mathbf{F}(\mathbf{U}_j^n)) + \Delta t\,\mathbf{S}_{j+\frac{1}{2}}^n$$

$$+\frac{1}{2}(\mathbf{U}_{j-\frac{1}{2}}^{n,+} - \mathbf{U}_{j+\frac{1}{2}}^{n,+} - \mathbf{U}_j^n + \mathbf{U}_{j+1}^n).$$

The sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ being an equilibrium solution we have

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n = \mathbf{U}_{j-\frac{1}{2}}^{n,+} \text{ and } \mathbf{U}_{j+1}^n = \mathbf{U}_{j+\frac{1}{2}}^{n,+},$$

and relation (3.17) follows. □

Let us now define the concept of well-balanced numerical scheme.

Definition 3.4. The Godunov-type scheme (3.10) is said to be well-balanced if and only if, for all sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ satisfying (3.17), the sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ is an equilibrium solution *i.e.* satisfies (3.18).

It turns out that relations (3.18) may be difficult to verify in practice. We are thus led to introduce a stronger (but easier to manipulate) notion of equilibrium solution[14]).

Definition 3.5. The sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ is said to be a strong equilibrium solution of the Godunov-type numerical scheme (3.10) if for all $j \in \mathbb{Z}$ and $t > 0$

$$\widetilde{\mathbf{W}}(x/t; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n) = \begin{cases} \mathbf{U}_j^n & x < 0, \\ \mathbf{U}_{j+1}^n & x > 0. \end{cases} \tag{3.21}$$

It is clear that (3.21) implies (3.18) so a strong equilibrium solution is an equilibrium solution in the sense of definition 3.3. To conclude this section, we define the corresponding notion of strongly well-balanced numerical scheme.

Definition 3.6. The Godunov-type numerical scheme (3.10) is said to be strongly well-balanced if and only if for all sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ satisfying (3.17), the sequence $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ is a strong equilibrium solution *i.e.* satisfies (3.21).

3.3. *The first example of the gas dynamics equations in Lagrangian coordinates*

We focus in this section on the gas dynamics equations (2.3) in Lagrangian coordinates with friction and gravity terms. For convenience, we repeat this PDE model here:

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m p = (g - \alpha\varphi(u)), \\ \partial_t e + \partial_m (pu) = (gu - \alpha\psi(u)), \end{cases} \tag{3.22}$$

which can be recast in the form (1.1) with the choice

$$\mathbf{U} = \begin{pmatrix} \tau \\ u \\ e \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} -u \\ p \\ pu \end{pmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ g - \alpha\varphi(u) \\ gu - \alpha\psi(u) \end{pmatrix} \tag{3.23}$$

with friction terms (2.2).

In the next paragraph we give a consistent and simple approximate Riemann solver introduced in Ref. 13 for Euler system with gravity ((3.22) with $\alpha = 0$). The corresponding *well-balanced* and *asymptotic preserving* properties are studied in subsections 3.3.2 and 3.3.3.

3.3.1. *Derivation of a simple approximate Riemann solver*

The proposed simple approximate Riemann solver (3.3) is made of three waves ($l = 3$) and two intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$. The two extreme waves propagate with speeds $\lambda_1 = -C$ and $\lambda_3 = C$ whereas the intermediate one is stationary ($\lambda_2 = 0$):

$$\widetilde{\mathbf{W}}(\frac{m}{t};\mathbf{U}_L,\mathbf{U}_R) = \begin{cases} \mathbf{U}_L, & \frac{m}{t} < -C, \\ \mathbf{U}_L^*, & -C < \frac{m}{t} < 0, \\ \mathbf{U}_R^*, & 0 < \frac{m}{t} < C, \\ \mathbf{U}_R, & \frac{m}{t} > C. \end{cases} \tag{3.24}$$

The parameter C is an approximation of the exact Lagrangian sound speed $C^{\mathcal{L}}$ associated with the acoustic waves of system (3.22) (recall that $(C^{\mathcal{L}})^2 = -\partial_\tau \tilde{p}(\tau,\eta)$, where the pressure law $p = \tilde{p}(\tau,\eta)$ is expressed in terms of (τ,η)). We will see further below that C has to be taken large enough with respect to the sound speed.
In order to define the intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$, we first write that (3.24) should be consistent with the integral form of (3.22). By Definition 3.1, there must exist a function $\widetilde{\mathbf{S}}(\xi,\tau;\mathbf{U}_L,\mathbf{U}_R)$ such that

$$\mathbf{F}(\mathbf{U}_R)-\mathbf{F}(\mathbf{U}_L)-\Delta x\ \widetilde{\mathbf{S}}(\Delta m,\Delta t;\mathbf{U}_L,\mathbf{U}_R) = -C(\mathbf{U}_L^*-\mathbf{U}_L)+C(\mathbf{U}_R-\mathbf{U}_R^*) \tag{3.25}$$

with

$$\lim_{\substack{\Delta m,\Delta t\to 0\\ \mathbf{U}_L,\mathbf{U}_R\to\mathbf{U}}} \widetilde{\mathbf{S}}(\Delta m,\Delta t;\mathbf{U}_L,\mathbf{U}_R) = \mathbf{S}(\mathbf{U}). \tag{3.26}$$

At this stage, we naturally choose $\widetilde{\mathbf{S}}$ and seek for $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$ such that (3.25) is true. It is reasonable to set

$$\widetilde{\mathbf{S}}(\Delta m,\Delta t;\mathbf{U}_L,\mathbf{U}_R) = \begin{pmatrix} 0 \\ g-\alpha\varphi(\tilde{u}) \\ g\tilde{u}-\alpha\psi(\tilde{u}) \end{pmatrix} \tag{3.27}$$

with $\tilde{u} = \tilde{u}(\Delta m,\Delta t;\mathbf{U}_L,\mathbf{U}_R)$ to be precised later on. Then, consistency relations (3.25) now read

$$\begin{cases} -\Delta u = C(\tau_L-\tau_L^*+\tau_R-\tau_R^*), \\ \Delta p-\Delta m(g-\alpha\varphi(\tilde{u})) = C(u_L-u_L^*+u_R-u_R^*), \\ \Delta(pu)-\Delta m(g\tilde{u}-\alpha\psi(\tilde{u})) = C(e_L-e_L^*+e_R-e_R^*), \end{cases} \tag{3.28}$$

where we have used the classical notation $\Delta X = X_R - X_L$ for each quantity X related to the fluid.

Let us now turn to the definition of the intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$. First, we note that the mass conservation equation in (3.22) does not contain any source term. So it is natural to impose the usual Rankine-Hugoniot jump relation associated with this first equation across each of the three waves of (3.24). We easily get

$$\begin{cases} u_L^* - C\tau_L^* = u_L - C\tau_L, \\ u_R^* = u_L^*, \\ u_R + C\tau_R = u_R^* + C\tau_R^*. \end{cases} \tag{3.29}$$

Then, the first relation in (3.28) becomes automatically satisfied whereas the second one writes

$$\Delta p - \Delta m(g - \alpha\varphi(\tilde{u})) = 2C(u_a - u^*),$$

where we have set $u^* = u_L^* = u_R^*$ and $X_a = \frac{1}{2}(X_L + X_R)$ for each pair (X_L, X_R). This relation, together with the first and third ones in (3.29), allows to obtain the following formulas for u^*, τ_L^* and τ_R^*:

$$u^* = u_a - \frac{1}{2C}\Delta P, \quad \Delta P = \Delta p - \Delta m(g - \alpha\varphi(\tilde{u})), \tag{3.30}$$

$$\begin{cases} \tau_L^* = \tau_L + \dfrac{1}{2C}\Delta u - \dfrac{1}{2C^2}\Delta P, \\ \tau_R^* = \tau_R + \dfrac{1}{2C}\Delta u + \dfrac{1}{2C^2}\Delta P. \end{cases} \tag{3.31}$$

Let us make a first point. At this stage, $\tilde{u}$ is still unknown, $u^* = u_L^* = u_R^*$, τ_L^* and τ_R^* are given and only depend on ΔP (which depends itself on $\tilde{u}$), and the first two consistency relations in (3.28) are satisfied. In order to complete the definition of the intermediate states it thus remains to specify e_L^* and e_R^*. These two quantities are related by a single compatibility relation, namely the third equation in (3.28), so that one has another degree of freedom in addition to $\tilde{u}$. Actually, $\tilde{u}$ will be defined in subsection 3.3.3 so as to get the *asymptotic preserving* property. To fix the other degree of freedom, a particular form for the numerical flux $\widetilde{\mathbf{G}}(\mathbf{U}_L, \mathbf{U}_R)$ is proposed. From (3.11) we have

$$\widetilde{\mathbf{G}}(\mathbf{U}_L, \mathbf{U}_R) = \frac{1}{2}\left\{\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R) - C(\mathbf{U}_L^* - \mathbf{U}_L + \mathbf{U}_R - \mathbf{U}_R^*)\right\}. \tag{3.32}$$

Using (3.30) and (3.31), the first component is given by

$$\frac{1}{2}\left\{-u_L - u_R - C(\tau_L^* - \tau_L + \tau_R - \tau_R^*)\right\} = \frac{1}{2}(-2u_a + \frac{1}{C}\Delta P) = -u^*,$$

the second by

$$\frac{1}{2}\left\{p_L + p_R - C(u^* - u_L + u_R - u^*)\right\} = p^*$$

with

$$p^* = p_a - \frac{C}{2}\Delta u, \tag{3.33}$$

and the third one writes

$$\frac{1}{2}\left\{p_L u_L + p_R u_R - C(e_L^* - e_L + e_R - e_R^*)\right\}.$$

We make our choice right now, assuming that the friction term is such that $a = 1$. Since the third component of the flux $\mathbf{F}(\mathbf{U})$ and the source term $\mathbf{S}(\mathbf{U})$ are both obtained by multiplying the second components by u, we propose to mimic this behavior at the numerical level by imposing that the third component of the numerical flux equals p^*u^*, namely

$$\frac{1}{2}\left\{p_L u_L + p_R u_R - C(e_L^* - e_L + e_R - e_R^*)\right\} = p^* u^*.$$

Combining this relation with the third consistency relation in (3.28) easily gives $e_L^* - e_L$ and $e_R^* - e_R$. More precisely

$$\begin{cases} e_L^* - e_L = \dfrac{1}{C}\left\{p_L u_L - p^* u^* + \dfrac{\Delta m}{2}(g\tilde{u} - \alpha\psi(\tilde{u}))\right\}, \\ \\ e_R^* - e_R = \dfrac{1}{C}\left\{p^* u^* - p_R u_R + \dfrac{\Delta m}{2}(g\tilde{u} - \alpha\psi(\tilde{u}))\right\}, \end{cases} \tag{3.34}$$

which completes the determination the intermediate states $\mathbf{U}_L^*$ and $\mathbf{U}_R^*$, to within the choice of $\tilde{u}$. To conclude, note that the proposed approximate Riemann solver, with $\widetilde{S}$ of the form (3.27), is consistent with the integral form of (3.22) provided that $\tilde{u}$ is such that

$$\lim_{\substack{\Delta m, \Delta t \to 0 \\ \mathbf{U}_L, \mathbf{U}_R \to \mathbf{U}}} \tilde{u}(\Delta m, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = u, \tag{3.35}$$

in order to recover (3.26).

3.3.2. *Well-balanced property of the scheme*

In this section we assess the well-balanced property of the Godunov-type scheme associated with the simple Riemann solver we have just derived. We note two given neighboring cell states by $\mathbf{U}_j^n = \mathbf{U}_L, \mathbf{U}_{j+1}^n = \mathbf{U}_R$.

Lemma 1. *Consider a discrete equilibrium solution satisfying (3.17) i.e.*

$$\frac{1}{\Delta m}(\boldsymbol{F}(\boldsymbol{U}_R) - \boldsymbol{F}(\boldsymbol{U}_L)) = \widetilde{\boldsymbol{S}}(\Delta m, \Delta t; \boldsymbol{U}_L, \boldsymbol{U}_R). \tag{3.36}$$

then, for the approximate Riemann solver (3.24), the intermediate states satisfy

$$\boldsymbol{U}_L^* = \boldsymbol{U}_L, \quad \boldsymbol{U}_R^* = \boldsymbol{U}_R.$$

Thus the discrete equilibrium is a strong equilibrium in the sense of Definition 5. Moreover, the velocity is constant $\tilde{u} = u^* = u_L = u_R$.

Proof. The discrete equilibrium condition (3.17) with $\mathbf{U}_j = \mathbf{U}_L$ and $\mathbf{U}_{j+1} = \mathbf{U}_R$ writes here in the form (3.36)

$$\frac{1}{\Delta m}(\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L)) = \widetilde{\mathbf{S}}(\Delta m, \Delta t; \mathbf{U}_L, \mathbf{U}_R),$$

that is, using the same notations as above,

$$\begin{cases} \Delta u = 0, \\ \\ \Delta p - \Delta m(g - \alpha\varphi(\tilde{u})) = 0, \\ \\ \Delta(pu) - \Delta m(g\tilde{u} - \alpha\psi(\tilde{u})) = 0. \end{cases} \tag{3.37}$$

The first two equalities give

$$\begin{cases} u_L = u_R = \overline{u}, \\ \\ \tilde{u} = \varphi^{-1}(\frac{1}{\alpha}(-\frac{\Delta p}{\Delta m} + g)), \end{cases}$$

that is, invoking (3.30), (3.31) and (3.33)

$$\begin{cases} u^* = u_a = \overline{u}, \\ \\ \tau_L^* = \tau_L, \quad \tau_R^* = \tau_R, \\ \\ p^* = p_a. \end{cases}$$

The third condition (3.37) then gives $\overline{u}\Delta p - \Delta m(g\tilde{u} - \alpha\psi(\tilde{u})) = 0$ so that

$$p_L u_L - p^* u^* + \frac{\Delta m}{2}(g\tilde{u} - \alpha\psi(\tilde{u})) = p^* u^* - p_R u_R + \frac{\Delta m}{2}(g\tilde{u} - \alpha\psi(\tilde{u}))$$

$$= -\frac{1}{2}(\overline{u}\Delta p - \Delta m(g\tilde{u} - \alpha\psi(\tilde{u}))) = 0$$

and from (3.34), we get: $e_L^* = e_L, \quad e_R^* = e_R$. Then, the discrete equilibrium condition (3.36) implies

$$\mathbf{U}_L^* = \mathbf{U}_L, \quad \mathbf{U}_R^* = \mathbf{U}_R.$$

which gives the result. □

We have thus also proved:

Proposition 3.4. *The Godunov-type scheme associated with the consistent and simple approximate Riemann solver defined by (3.24)-(3.30)-(3.31)-(3.34) is a strongly well-balanced scheme for system (3.22) in the sense of Definition 3.6.*

Let us precise the stationary solutions computed by the scheme. They are naturally such that $u = \overline{u}$ is constant. Now, from the study of stationary solutions for the continuous system, either $\overline{u} = 0$, or $\overline{u} \neq 0$ but in any case $\psi(\overline{u}) = \overline{u}\varphi(\overline{u})$ which we assume for the lines below. Do we have the discrete analog of (2.7) when $\overline{u} = 0$ or (2.8) when $\overline{u} \neq 0$?

The second and the third equations of (3.37) successively give

$$\overline{u}\frac{\Delta p}{\Delta m} = \overline{u}(g - \alpha\varphi(\tilde{u})) = \tilde{u}(g - \alpha\varphi(\tilde{u})),$$

which yields $\tilde{u} = \overline{u}$ or $\alpha\varphi(\tilde{u}) = g$. The last solution may occur in the particular case where $\tilde{u} = \varphi^{-1}(\frac{g}{\alpha})$, and it implies $\frac{\Delta p}{\Delta m} = 0$, thus $p_L = p_R$ is constant, and we can have $\overline{u} = \varphi^{-1}(\frac{g}{\alpha})$ too. If $\tilde{u} \neq \overline{u}$, the consistency condition (3.35) requires $\tilde{u} \to \overline{u}$, so that $\overline{u}$ should be equal to $\varphi^{-1}(\frac{g}{\alpha})$ and this is naturally coherent with (2.8) when the pressure is constant, so that $\tilde{u} \neq \overline{u}$ is indeed inconsistent. In all other cases, we have $\tilde{u} = \overline{u}$, and the consistency condition (3.35) is automatically satisfied.

Remark 3.4. The results of Proposition 3.4 involve the definition (3.36) of a discrete equilibrium, and this definition in turn is linked to the (quite natural) choice of the source term in the form $\mathbf{S}(\tilde{u})$ as given by (3.27). Let us just remark that in other situations, one might think that the source term is defined at $\mathcal{O}(\Delta x)$, for instance setting $\widetilde{S}(\Delta m, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \mathbf{S}(\tilde{u}) + \Delta x$, would still lead to a consistent scheme, but then, for such a definition, the discrete equilibrium should also change and be defined with $\mathbf{S}(\tilde{u})$, which gives (3.36) at the order $\mathcal{O}(\Delta x)$. This is to illustrate that the notion of discrete equilibrium is scheme dependent (at the order $\mathcal{O}(\Delta x)$). Note also that the choice of $\mathcal{O}(\Delta x)$ terms depending on the parameter α might reveal itself particularly inadequate for deriving asymptotic preserving properties. □

The case of stationary solutions is of course peculiar and the well balanced property is obtained as soon as $\tilde{u}$ satisfies $\tilde{u} = u$ once $u_L = u_R$, which holds if we take $\tilde{u} = u^*$ but also for $\tilde{u} = u_a$ or for any usual average of the two values. The asymptotic preserving property of the scheme requires a specific choice for $\tilde{u}$, or at least is not compatible with all choices as we will see below.

3.3.3. *Asymptotic preserving property of the scheme*

The asymptotic preserving property of the scheme requires a specific choice for $\tilde{u}$. From now on, we choose

$$\tilde{u} = u^* \tag{3.38}$$

and postpone the discussion on this choice in Remark 3.5 below. The choice (3.38) together with (3.30) makes the computation of u^* implicit in the sense that the following (generally) nonlinear scalar equation has to be solved :

$$u^* + \frac{\alpha\Delta m}{2C}\varphi(u^*) = u_a - \frac{1}{2C}(\Delta p - \Delta m\, g). \tag{3.39}$$

Since $u \mapsto u + \frac{\alpha\Delta m}{2C}\varphi(u)$ is a strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$, (3.39) admits a unique solution $u^*(\Delta m; \mathbf{U}_L, \mathbf{U}_R)$ [a] that we write

$$u^*(\Delta m; \mathbf{U}_L, \mathbf{U}_R) = \Phi(u_a - \frac{1}{2C}(\Delta p - \Delta m\, g); \frac{\alpha\Delta m}{2C}). \tag{3.40}$$

Importantly, we note that (3.35) holds true by continuity arguments. Setting

$$u_{j+\frac{1}{2}} = u^*(\Delta m; \mathbf{U}_j, \mathbf{U}_{j+1}), \quad p_{j+\frac{1}{2}} = p^*(\mathbf{U}_j, \mathbf{U}_{j+1})$$

and skipping the time superscripts the variable to soften the notations (the updated value beeing overlined), the numerical scheme writes

$$\begin{cases} \bar{\tau}_j = \tau_j + \dfrac{\Delta t}{\Delta m}(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}), \\ \bar{u}_j = u_j - \dfrac{\Delta t}{\Delta m}(p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}) + \Delta t\Big(g - \dfrac{\alpha}{2}\big(\varphi(u_{j-\frac{1}{2}}) + \varphi(u_{j+\frac{1}{2}})\big)\Big), \\ \bar{e}_j = e_j - \dfrac{\Delta t}{\Delta m}(p_{j+\frac{1}{2}}u_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}u_{j-\frac{1}{2}}) \\ \qquad + \dfrac{\Delta t}{2}\Big(g(u_{j-\frac{1}{2}} + u_{j+\frac{1}{2}}) - \alpha\big(\psi(u_{j-\frac{1}{2}}) + \psi(u_{j+\frac{1}{2}})\big)\Big) \end{cases} \tag{3.41}$$

with, using (3.40),

$$\begin{cases} u_{j+\frac{1}{2}} = \Phi\Big(\dfrac{1}{2}\big(u_j + u_{j+1} - \dfrac{1}{C}(p_{j+1} - p_j - \Delta m\, g)\big); \dfrac{\alpha\Delta m}{2C}\Big), \\ p_{j+\frac{1}{2}} = \dfrac{1}{2}\left\{p_j + p_{j+1} - C(u_{j+1} - u_j)\right\}. \end{cases} \tag{3.42}$$

Now we make the change of variable

$$\Delta t = \beta\Delta s, \quad v = \beta u, \quad \beta = \alpha^{\frac{1}{\chi+1}} \tag{3.43}$$

so that if $v_L = \beta u_L$ and $v_R = \beta u_R$, (3.39) becomes

$$u^* + \frac{\Delta m}{2C}\varphi(\beta u^*) = \frac{1}{\beta}v_a - \frac{1}{2C}(\Delta p - \Delta m\, g).$$

When β goes to $+\infty$, one has $\beta u^* \sim \varphi^{-1}(-\frac{\Delta p}{\Delta x} + g)$ that is

$$u^* = \frac{1}{\beta}\varphi^{-1}(-\frac{\Delta p}{\Delta m} + g) + o(\frac{1}{\beta}), \quad \beta \to +\infty.$$

It is relevant to set

$$u^* = \frac{1}{\beta}v^*, \quad \text{then } v^* = \varphi^{-1}(-\frac{\Delta p}{\Delta m} + g) + \mathcal{O}(\frac{1}{\beta}) \tag{3.44}$$

[a] In fact, u^* does not depend on Δt.

and the scheme becomes

$$\begin{cases} \bar{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}), \\ \dfrac{1}{\beta}\bar{v}_j = \dfrac{1}{\beta}v_j - \dfrac{\beta\Delta s}{\Delta x}(p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}) + \beta\Delta s\Big(g - \dfrac{1}{2}\big(\varphi(v_{j-\frac{1}{2}}) + \varphi(v_{j+\frac{1}{2}})\big)\Big), \\ \bar{\varepsilon}_j + \dfrac{1}{2\beta^2}\bar{v}_j^2 = \varepsilon_j + \dfrac{1}{2\beta^2}v_j^2 - \dfrac{\Delta s}{\Delta m}(p_{j+\frac{1}{2}}v_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}v_{j-\frac{1}{2}}) \\ \qquad + \dfrac{\Delta s}{2}\Big(g(v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}) - \big(\psi(v_{j-\frac{1}{2}}) + \psi(v_{j+\frac{1}{2}})\big)\Big) \end{cases}$$

with

$$\begin{cases} \dfrac{1}{\beta}v_{j+\frac{1}{2}} + \dfrac{\Delta m}{2C}\varphi(v_{j+\frac{1}{2}}) = \dfrac{1}{2\beta}(v_j + v_{j+1}) - \dfrac{1}{2C}(p_{j+1} - p_j - \Delta m\, g), \\ p_{j+\frac{1}{2}} = \dfrac{1}{2}\left\{p_j + p_{j+1} - \dfrac{C}{\beta}(v_{j+1} - v_j)\right\}. \end{cases}$$

In the limit β goes to $+\infty$, the numerical scheme tends to

$$\begin{cases} \bar{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}), \\ \dfrac{1}{\Delta m}(p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}) = g - \dfrac{1}{2}\big(\varphi(v_{j-\frac{1}{2}}) + \varphi(v_{j+\frac{1}{2}})\big), \\ \bar{\varepsilon}_j = \varepsilon_j - \dfrac{\Delta s}{\Delta m}(p_{j+\frac{1}{2}}v_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}v_{j-\frac{1}{2}}) \\ + \dfrac{\Delta s}{2}\Big(g(v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}) - \big(\psi(v_{j-\frac{1}{2}}) + \psi(v_{j+\frac{1}{2}})\big)\Big) \end{cases} \tag{3.45}$$

with

$$\begin{cases} v_{j+\frac{1}{2}} = \varphi^{-1}(-\dfrac{p_{j+1} - p_j}{\Delta m} + g), \\ p_{j+\frac{1}{2}} = \dfrac{1}{2}(p_j + p_{j+1}). \end{cases} \tag{3.46}$$

In particular, the pair (τ, ε) evolves according to

$$\begin{cases} \bar{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}), \\ \bar{\varepsilon}_j = \varepsilon_j - \dfrac{\Delta s}{\Delta m}(p_{j+\frac{1}{2}}v_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}v_{j-\frac{1}{2}}) \\ + \dfrac{\Delta s}{2}\Big(g(v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}) - \big(\psi(v_{j-\frac{1}{2}}) + \psi(v_{j+\frac{1}{2}})\big)\Big) \end{cases} \tag{3.47}$$

which together with the relations (3.46) and $p_j = p(\tau_j, \varepsilon_j)$ is indeed a consistent explicit numerical scheme of the asymptotic system

$$\begin{cases} \partial_s \tau - \partial_m v = 0, \\ \\ \partial_s \varepsilon + \partial_m (pv) = gv - \psi(v) \end{cases} \tag{3.48}$$

with

$$\begin{cases} v = \varphi^{-1}(g - \partial_m p), \\ \\ p = p(\tau, \varepsilon). \end{cases} \tag{3.49}$$

To sum up, we have shown:

Proposition 3.5. *The Godunov-type scheme associated with the consistent and simple approximate Riemann solver defined by (3.24)-(3.30)-(3.31)-(3.34) and $\tilde{u}$ given by (3.38)-(3.39) is asymptotic preserving for system (3.22).*

It is *asymptotic preserving* in the sense that, after performing the same scaling (3.43) as the one done to obtain the asymptotic system (2.16), it becomes a consistent scheme for the asymptotic system. We can say shortly that it preserves the asymptotic behavior of the solutions of (3.22) in the limit $\alpha \to +\infty$.

Remark 3.5. If we take for instance $\tilde{u} = u_a$ instead of $\tilde{u} = u^*$, it is worth noticing that the asymptotic preserving property is lost. Indeed, we deduce from (3.30) the relation

$$u^* = u_a - \frac{1}{2C}\Big(\Delta p - \Delta m\big(g - \alpha\varphi(\tilde{u})\big)\Big), \tag{3.50}$$

and applying the change of variable (3.43) we get if $\tilde{u} = u_a$,

$$u^* = \frac{1}{\beta} v_a - \frac{1}{2C}\Big(\Delta p - \Delta m\big(g - \varphi(v_a)\big)\Big).$$

We then observe that setting $u^* = \frac{1}{\beta} v^*$ is no longer relevant because of the terms of different orders involving v_a and the previous asymptotic analysis fails.

In fact, it is natural to require that the change of variables $v = \beta u$ be relevant for all velocities under consideration u_a, u^* and $\tilde{u}$. Let us start from (3.50) which gives the relation between $u^*(\tilde{u})$ in tems of $\tilde{u}$, Since the change of variables satisfies $\alpha\varphi(u) = \varphi(\beta u)$, we get

$$\frac{v^*}{\beta} = \frac{v_a}{\beta} - \frac{1}{2C}(\Delta p - \Delta m(g - \varphi(\tilde{v}))$$

which implies

$$\varphi(\tilde{v}) = g - \frac{\Delta p}{\Delta m} + \mathcal{O}(\frac{1}{\beta}).$$

This relation, which is the discrete analog of the identity we got for v in (2.22), is indeed obtained in (3.44) for the choice $\tilde{u} = u^*$.

We can precise the statement and prove that the choice is unique up to terms of order $o(\frac{1}{\beta\Delta m})$. Just for the following lines, note by $\tilde{u}_0 = u_0^*$ the unique solution of (3.30) such that $u^*(\tilde{u}) = \tilde{u}$, defined in (3.40). Here we have dropped the notation Φ in (3.40) and noted $u^*(\tilde{u}) = u_a - \frac{1}{2C}(\Delta p - \Delta m(g - \alpha\varphi(\tilde{u}))$, thus $\tilde{u}_0$ satisfies

$$\tilde{u}_0 = u_0^*(\Delta m; \mathbf{U}_L, \mathbf{U}_R) = \Phi(u_a - \frac{1}{2C}(\Delta p - \Delta m\, g); \frac{\alpha\Delta m}{2C}).$$

Then,

$$u^*(\tilde{u}) - \tilde{u}_0 = \alpha\Delta m(\varphi(\tilde{u}) - \varphi(\tilde{u}_0))$$

and after scaling, with $v = \beta u$, since $\varphi(\beta u) = \alpha\varphi(u)$ it yields

$$v^*(\tilde{v}) - \tilde{v}_0 = \beta\Delta m(\varphi(\tilde{v}) - \varphi(\tilde{v}_0)).$$

Now the expected asymptotic behavior for $\tilde{v}_0$ is

$$\tilde{v}_0 \to v^A \equiv \varphi^{-1}(g - \frac{\Delta m}{\Delta p}) \text{ as } \beta \to \infty.$$

Assume for simplicity a linear friction term (but it can be extended to the general case), then $v^A = g - \frac{\Delta m}{\Delta p}$. We can easily check from (3.39) that, indeed,

$$\tilde{v}_0 = v^A + C(v_a - v^A)(\frac{1}{\beta\Delta m}) + o(\frac{1}{\beta\Delta m}).$$

Now, since

$$v^*(\tilde{v}) - \tilde{v}_0 = \beta\Delta m(\tilde{v} - \tilde{v}_0),$$

if we do not take $\tilde{v} = \tilde{v}_0$ but say add a first order correction term $\tilde{v} = \tilde{v}_0 + \frac{1}{\beta\Delta m}\tilde{v}_0^1 + o(\frac{1}{\beta\Delta m})$ for some $\tilde{v}_0^1$, then $v^*(\tilde{v}) - \tilde{v}_0 = \tilde{v}_0^1 + \mathcal{O}(\frac{1}{\beta\Delta m})$ and

$$v^*(\tilde{v}) = v^A + \tilde{v}_0^1 + \mathcal{O}(\frac{1}{\beta\Delta m}).$$

We can only take a second order correction term $\tilde{v}_0^1 = \mathcal{O}(\frac{1}{\beta\Delta m})$ if we want to preserve the asymptotic behavior, $v^*(\tilde{v}) = v^A + \mathcal{O}(\frac{1}{\beta\Delta m})$.

From a heuristic point of view, this choice $\tilde{u} = u^*$ for the velocity of the source term $\tilde{u}$ is natural; u^* is the common speed of the intermediate states of the Riemann solver (in $-C < m/t < C$) and we are considering the large time behavior $t = \beta s$, so that considering the value inside the fan for the asymptotic velocity is judicious. □

4. Application to the gas dynamics equations in Eulerian coordinates

We have designed in the previous section a well-balanced and asymptotic preserving scheme for the gas dynamics equations with friction and gravity terms in Lagrangian coordinates. It is based on the definition of a suitable simple approximate Riemann solver. Our objective in this section is to extend it to the Eulerian formulation. At

the continuous level, the equivalence between the two formulations relies ont the results of Ref. 32. With this in mind, we first recall in subsection 4.1 how to match a simple approximate Riemann solver in Eulerian coordinates with a natural one in Lagrangian coordinates (following Gallice[15]). Then, we use this general procedure in subsection 4.3 to devise an asymptotic preserving scheme for gas dynamics equations with friction and gravity terms in Eulerian coordinates.

4.1. *From Lagrangian to Eulerian coordinates*

We begin by some preliminaries and notations. Let first be given a system of partial differential equations in Eulerian coordinates of the following form

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \\ \partial_t(\varrho\Psi) + \partial_x(\varrho\Psi u + \mathbf{f}(\varrho,\Psi)) = \varrho\mathbf{s}(\varrho,\Psi), \end{cases} \tag{4.1}$$

where we have put aside the equation of conservation of mass, which has no source term, the convection part for the vector $\varrho\Psi$ of the other conservative variables, say $\Psi \in \mathbb{R}^{n-1}$, has been singled out ($n = 3$ for system (2.1) but we may consider more general systems), and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$ is the remaining part of the flux. We will write for short

$$\partial_t \mathbf{U}^E + \partial_x \mathbf{F}^E(\mathbf{U}^E) = \mathbf{S}^E(\mathbf{U}^E) \tag{4.2}$$

with

$$\mathbf{U}^E = \begin{pmatrix} \varrho \\ \varrho\Psi \end{pmatrix}, \mathbf{F}^E(\mathbf{U}^E) = \begin{pmatrix} \varrho u \\ \varrho\Psi u + \mathbf{f}(\varrho,\Psi) \end{pmatrix}, \mathbf{S}^E(\mathbf{U}^E) = \begin{pmatrix} 0 \\ \varrho\mathbf{s}(\varrho,\Psi) \end{pmatrix}. \tag{4.3}$$

System (4.3) is supplemented with an entropy inequality

$$\partial_t \eta^E(\mathbf{U}^E) + \partial_x \mathcal{Q}^E(\mathbf{U}^E) \leq \sigma^E(\mathbf{U}^E) = (\eta^E)'(\mathbf{U}^E) \cdot \mathbf{S}^E(\mathbf{U}^E), \tag{4.4}$$

where $\eta^E : \mathbb{R}^n \mapsto \mathbb{R}$ is the (convex) entropy with entropy flux $\mathcal{Q}^E$. In Lagrangian coordinates, this system writes, setting $\tau = \frac{1}{\varrho}$, $m = \int \varrho dx$,

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \\ \partial_t \Psi + \partial_m \mathbf{f}(\frac{1}{\tau},\Psi) = \mathbf{s}(\frac{1}{\tau},\Psi), \end{cases} \tag{4.5}$$

or equivalently

$$\partial_t \mathbf{U}^{\mathcal{L}} + \partial_m \mathbf{F}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) = \mathbf{S}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) \tag{4.6}$$

with

$$\mathbf{U}^{\mathcal{L}} = \begin{pmatrix} \tau \\ \Psi \end{pmatrix}, \quad \mathbf{F}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) = \begin{pmatrix} -u \\ \mathbf{f}(\frac{1}{\tau},\Psi) \end{pmatrix}, \quad \mathbf{S}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) = \begin{pmatrix} 0 \\ \mathbf{s}(\frac{1}{\tau},\Psi) \end{pmatrix}, \tag{4.7}$$

and

$$\partial_t \eta^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) + \partial_x \mathcal{Q}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) \leq \sigma^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}) \tag{4.8}$$

where

$$\eta^E = \varrho\eta^{\mathcal{L}}, \quad \mathcal{Q}^E = \eta^{\mathcal{L}}\, u + \mathcal{Q}^{\mathcal{L}} \quad \text{and} \quad \sigma^E = \varrho\sigma^{\mathcal{L}}.$$

In fact $\mathcal{Q}^{\mathcal{L}} = 0$ for fluid systems and in particular for (2.3).

Note that the mapping $\mathbf{U}^{\mathcal{L}} \mapsto \mathbf{U}^E$ defines an admissible change of variables in $\mathbb{R}^n$ under the natural assumption $\varrho > 0$. In addition, easy calculations show that the eigenvalues $\lambda_k(\mathbf{U}^E), 1 \leq k \leq n$, of the Jacobian matrix of $\mathbf{F}^E(\mathbf{U}^E)$ are related to the eigenvalues $\mu_k, 1 \leq k \leq n$, of the Jacobian matrix of $\mathbf{F}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}})$ by the following relation:

$$\lambda_k(\mathbf{U}^E) = u + \tau\mu_k(\mathbf{U}^{\mathcal{L}}). \tag{4.9}$$

4.2. *Simple approximate Riemann solvers*

We now consider a simple approximate Riemann solver in Lagrangian coordinates associated with (4.5), that is

$$\widetilde{\mathbf{W}}^{\mathcal{L}}(\frac{m}{t}; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}}) = \begin{cases} \mathbf{U}_1^{\mathcal{L}} = \mathbf{U}_L^{\mathcal{L}}, & \frac{m}{t} < \mu_1, \\ \mathbf{U}_k^{\mathcal{L}}, & \mu_{k-1} < \frac{m}{t} < \mu_k, \quad k = 2, .., l, \\ \mathbf{U}_{l+1}^{\mathcal{L}} = \mathbf{U}_R^{\mathcal{L}}, & \frac{m}{t} > \mu_l, \end{cases} \tag{4.10}$$

where l, the number of approximate waves, satisfies $l \leq n$ (and again we use the same notation μ for the velocities and the eigenvalues of the Jacobian). We assume that the Rankine-Hugoniot relation associated with the first equation in (4.5) is satisfied across each approximate wave, which implies

$$u_k + \mu_k\tau_k = u_{k+1} + \mu_k\tau_{k+1}, \quad \text{for all} \quad k = 1, .., l. \tag{4.11}$$

Recall that this mass conservation property has already been imposed in subsection 3.3.1 (see relations (3.29)).

We then define the following natural simple approximate Riemann solver in Eulerian coordinates

$$\widetilde{\mathbf{W}}^E(\frac{x}{t}; \mathbf{U}_L^E, \mathbf{U}_R^E) = \begin{cases} \mathbf{U}_1^E = \mathbf{U}_L^E = \mathbf{U}^E(\mathbf{U}_L^{\mathcal{L}}), & \frac{x}{t} < \lambda_1, \\ \mathbf{U}_k^E = \mathbf{U}^E(\mathbf{U}_k^{\mathcal{L}}), & \lambda_{k-1} < \frac{x}{t} < \lambda_k, \quad k = 2, .., l, \\ \mathbf{U}_{l+1}^E = \mathbf{U}_R^E = \mathbf{U}^E(\mathbf{U}_R^{\mathcal{L}}), & \frac{x}{t} > \lambda_l \end{cases} \tag{4.12}$$

with

$$\lambda_k = u_k + \mu_k\tau_k = u_{k+1} + \mu_k\tau_{k+1}, \quad k = 1, .., l. \tag{4.13}$$

Assume now that (4.10) is consistent with the integral form of (4.6) in the sense of Definition 3.1, that is there exists a function $\widetilde{\mathbf{S}}^{\mathcal{L}}(\xi, \tau; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}})$ such that for $\Delta t/\Delta m$

satisfying the CFL condition (3.4) one has

$$\mathbf{F}^{\mathcal{L}}(\mathbf{U}_R^{\mathcal{L}}) - \mathbf{F}^{\mathcal{L}}(\mathbf{U}_L^{\mathcal{L}}) - \Delta m\, \widetilde{\mathbf{S}}^{\mathcal{L}}(\Delta m, \Delta t; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}}) = \sum_{k=1}^{l} \mu_k (\mathbf{U}_{k+1}^{\mathcal{L}} - \mathbf{U}_k^{\mathcal{L}}) \tag{4.14}$$

with

$$\lim_{\substack{\Delta m, \Delta t \to 0 \\ \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}} \to \mathbf{U}^{\mathcal{L}}}} \widetilde{\mathbf{S}}^{\mathcal{L}}(\Delta m, \Delta t; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}}) = \mathbf{S}^{\mathcal{L}}(\mathbf{U}^{\mathcal{L}}). \tag{4.15}$$

Easy calculations not reported here (see for instance Ref. 14) show that the validity of (4.14) is equivalent to

$$\mathbf{F}^{E}(\mathbf{U}_R^{E}) - \mathbf{F}^{E}(\mathbf{U}_L^{E}) - \Delta x\, \widetilde{\mathbf{S}}^{E}(\Delta x, \Delta t; \mathbf{U}_L^{E}, \mathbf{U}_R^{E}) = \sum_{k=1}^{l} \lambda_k (\mathbf{U}_{k+1}^{E} - \mathbf{U}_k^{E}) \tag{4.16}$$

where we have set $\Delta x = \frac{\Delta m}{\varrho^*(\mathbf{U}_L^E, \mathbf{U}_R^E)}$, for some positive density ϱ^* to be prescribed, and

$$\widetilde{\mathbf{S}}^{E}(\Delta x, \Delta t; \mathbf{U}_L^{E}, \mathbf{U}_R^{E}) = \varrho^*(\mathbf{U}_L^{E}, \mathbf{U}_R^{E})\, \widetilde{\mathbf{S}}^{\mathcal{L}}(\Delta m, \Delta t; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}}). \tag{4.17}$$

Provided that $\varrho^*(\mathbf{U}_L^E, \mathbf{U}_R^E)$ is such that

$$\lim_{\mathbf{U}_L^E, \mathbf{U}_R^E \to \mathbf{U}^E} \varrho^*(\mathbf{U}_L^E, \mathbf{U}_R^E) = \varrho(\mathbf{U}^E), \tag{4.18}$$

it is then clear that the simple approximate Riemann solver (4.12) in Eulerian coordinates is consistent with the integral form of (4.2) in the sense of Definition 3.1. A natural choice is provided by

$$\varrho^*(\mathbf{U}_L^E, \mathbf{U}_R^E) = \frac{1}{2}(\varrho_L^E + \varrho_R^E).$$

Remark 4.1. The CFL condition (3.4) obviously changes when we switch from Lagrangian to Eulerian coordinates. However the validity of (4.14) for the proposed approximate Riemann solver in subsection 3.3.1 is valid for any Δt and Δm. Then, the equivalence between (4.14) and (4.16) is actually sufficient to prove the consistency of the simple approximate Riemann solver (4.12) with the integral form of (4.2). □

As far as the consistency with the integral form of the entropy inequality is concerned, we easily prove (see again Ref. 1315) that the following two properties are equivalent, namely

$$\mathcal{Q}^{\mathcal{L}}(\mathbf{U}_R^{\mathcal{L}}) - \mathcal{Q}^{\mathcal{L}}(\mathbf{U}_L^{\mathcal{L}}) - \Delta m\, \tilde{\sigma}^{\mathcal{L}}(\Delta m, \Delta t; \mathbf{U}_L^{\mathcal{L}}, \mathbf{U}_R^{\mathcal{L}}) \leq \sum_{k=1}^{l} \mu_k (\eta^{\mathcal{L}}(\mathbf{U}_{k+1}^{\mathcal{L}}) - \eta^{\mathcal{L}}(\mathbf{U}_k^{\mathcal{L}})) \tag{4.19}$$

and

$$\mathcal{Q}^E(\mathbf{U}_R^E) - \mathcal{Q}^E(\mathbf{U}_L^E) - \Delta x\ \tilde{\sigma}^E(\Delta x, \Delta t; \mathbf{U}_L^E, \mathbf{U}_R^E) \leq \sum_{k=1}^{l} \lambda_k(\eta^E((\mathbf{U}_{k+1}^E) - \eta^E(\mathbf{U}_k^E)) \tag{4.20}$$

where we have set again $\Delta x = \frac{\Delta m}{\rho^*(\mathbf{U}_L^E, \mathbf{U}_R^E)}$, and

$$\tilde{\sigma}^E(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \rho^*(\mathbf{U}_L^E, \mathbf{U}_R^E)\ \tilde{\sigma}^{\mathcal{L}}(\Delta m, \Delta t; \mathbf{V}_L, \mathbf{V}_R). \tag{4.21}$$

In other words and in agreement with Definition 3.2, the simple approximate Riemann solver (4.12) in Eulerian coordinates is consistent with the integral form of the entropy inequality (4.4) as soon as the simple approximate Riemann solver (4.10) is consistent with the integral form of (4.8). Note that this result again relies on the validity of the Rankine-Hugoniot relations (4.11), together with the consistency condition (4.18).

Before we end the section let us say a word about the well-blanced character of the scheme. For Euler system, we have seen that interesting equilibria are $u = 0, \partial_x p = \varrho g$. If we define discrete equilibria by $u = 0$, and $\frac{\Delta p}{\Delta x} = \varrho^* g$, it can be writen $\frac{\Delta p}{\Delta m} = g$, then, if we take $u_L = u_R = 0$, $\tilde{u} = u^*$, (3.39) gives $u^* = 0, \tau_L^* = \tau_L, \tau_R^* = \tau_R$ and discrete equilibria are preserved, the scheme is strongly well-balanced.

4.3. *An asymptotic preserving scheme for the gas dynamics equations in Eulerian coordinates*

The framework of the previous subsection is now used to design a simple approximate Riemann solver for the gas dynamics equations with friction and gravity terms in Eulerian coordinates (2.1). The corresponding Godunov-type numerical scheme is shown to be asymptotic preserving.
The system writes

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0, \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \partial_t \varrho e + \partial_x(\varrho e + p)u = \varrho(gu - \alpha\psi(u)), \end{cases} \tag{4.22}$$

which corresponds in (4.1) to the choice $\Psi = (u, e)^T$, and $\mathbf{f}(\varrho, \Psi)) = (p, pu)^T$, $\mathbf{s}(\varrho, \Psi)) = (g - \alpha\varphi(u), gu - \alpha\psi(u))^T$ or

$$\mathbf{U} = \begin{pmatrix} \varrho \\ \varrho u \\ \varrho e \end{pmatrix}, \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \varrho u \\ \varrho u^2 + p \\ (\varrho e + p)u \end{pmatrix}, \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ \varrho\big(g - \alpha\varphi(u)\big) \\ \varrho\big(gu - \alpha\psi(u)\big) \end{pmatrix}. \tag{4.23}$$

According to the subsection 3.3.1, we then propose the following simple approximate Riemann solver in Eulerian coordinates :

$$\widetilde{\mathbf{W}}(\frac{x}{t};\mathbf{U}_L,\mathbf{U}_R)=\begin{cases}\mathbf{U}_L, & \frac{x}{t}<\lambda_1,\\ \mathbf{U}_L^*, & \lambda_1<\frac{x}{t}<\lambda_2,\\ \mathbf{U}_R^*, & \lambda_2<\frac{x}{t}<\lambda_3,\\ \mathbf{U}_R, & \frac{x}{t}>\lambda_3\end{cases} \tag{4.24}$$

with

$$\begin{cases}\lambda_1=u_L-C\tau_L=u^*-C\tau_L^*,\\ \lambda_2=u^*,\\ \lambda_3=u_R+C\tau_R=u^*+C\tau_R^*.\end{cases} \tag{4.25}$$

We recall that u^*, τ_L^* and τ_R^* solve (3.30)-(3.31) with here $\Delta m=\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\Delta x$, while e_L^* and e_R^* are given by (3.34) with p^* defined by (3.33). Setting

$$\widetilde{\mathbf{S}}(\Delta x,\Delta t;\mathbf{U}_L,\mathbf{U}_R)=\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\begin{pmatrix}0\\ g-\alpha\varphi(\tilde{u})\\ g\tilde{u}-\alpha\psi(\tilde{u})\end{pmatrix}, \tag{4.26}$$

with (4.18) assumed to be valid, the simple Riemann solver (4.24) is consistent with the integral form of (4.22).

Let us now pass to the asymptotic behavior of the scheme when α goes to infinity. As in subsection 3.3.1 we choose $\tilde{u}=u^*$, leading to the nonlinear scalar equation (3.39) for the actual calculation of u^*. We can state the following result, the proof of which follows that of the Lagrangian case in section 3.3.3, only with more technical points which are detailed in the apppendix:

Proposition 4.1. *The Godunov-type scheme associated with the consistent and simple approximate Riemann solver defined by (4.24)-(4.25)-(3.30)-(3.31)-(3.33)-(3.34) and $\tilde{u}$ given by (3.38)-(3.39) is asymptotic preserving in the sense that it preserves the asymptotic behavior of the solutions of (4.22) in the limit $\alpha\to+\infty$.*

5. Another approach

We consider again 1D Euler system with friction (2.1) which we rewrite for convenience:

$$\begin{cases}\partial_t\varrho+\partial_x(\varrho u)=0,\\ \partial_t(\varrho u)+\partial_x(\varrho u^2+p)=\varrho g-\alpha\varrho\varphi(u),\\ \partial_t(\varrho e)+\partial_x((\varrho e+p)u)=\varrho gu-\alpha\varrho\psi(u).\end{cases} \tag{5.1}$$

In this section, we will extend to system (5.1) both Cargo-Leroux' approach developped in Ref. 9 for Euler system with gravity (system (5.1) with $\alpha = 0$) for which only well-balanced properties were derived, and a relaxation scheme defined previously in Ref. 1 for Euler system (without source, $\alpha = g = 0$). Both ingredients are important to understand the properties of the resulting scheme which happens to coincide with the scheme previously derived: Cargo-Leroux' approach first transforms source terms in differential terms, and the relaxation approach gives an interpretation in terms of solutions of a PDE system. It is simpler to argue at the continuous level (with continuous variables) than with discretized data as we will see.

5.1. *Cargo-LeRoux's approach*

The idea in Ref. 9 is to transform the source in a differential term so that the system becomes *homogeneous*, which can be performed by introducing a 'potential' q such that

$$\begin{cases} \partial_x q = \varrho \\ \partial_t q = -\varrho u \end{cases} \tag{5.2}$$

and by writing (5.1) in an augmented non-conservative form

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) - (g - \alpha\varphi(u))\partial_x q = 0, \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) - (gu - \alpha\psi(u))\partial_x q = 0, \\ \partial_t(\varrho q) + \partial_x(\varrho u q) = 0. \end{cases} \tag{5.3}$$

The set of states for (5.3) is $\Omega_q^{Euler} = \{(\varrho, \varrho u, E = \varrho e, q); \varrho > 0, u \in \mathbb{R}, e - u^2/2 > 0, q \in \mathbb{R}\}$.

Remark 5.1. Note that Ref. 9 considers the case $\alpha = 0$ and thus q is rather defined as $\partial_x q = \varrho g$ (hydrostatic pressure), then $\partial_x q = -\varrho g u$. In that case, it is natural to define a 'new' pressure $\pi = p - q$ and a 'new' energy $\varrho\mathcal{E} = \varrho e + q$; the formulation is then conservative while our system has non-conservative terms. □

Let us first study the properties of system (5.3).

Lemma 2. *System (5.3) has four real eigenvalues $u - c, u, u, u + c$, where c is the usual sound speed ($c^2 = \frac{\partial p}{\partial \varrho}(\varrho, \eta)$) and it has a basis of eigenvectors if and only if*

$$\psi(u) = u\varphi(u) \tag{5.4}$$

or for the state $u = 0$. The first and last characteristic fields associated to $u \pm c$ are GNL, while the characteristic field associated to u is LD.

Proof. We write (5.3) in quasilinear form, and with the choice of variable $\mathbf{U} = (\varrho, \varrho u, \varrho e, q)^T$, the Jacobian matrix is

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} A^{Euler} & B_q \\ 0 & u \end{pmatrix} \tag{5.5}$$

where we have denoted by A^{Euler} the usual Jacobian 3×3 matrix of Euler system (in conservative variables $\mathbf{u} = (\varrho, \varrho u, \varrho e)^T$) and B_q is the column vector $B_q = (0, -(g - \alpha\varphi(u)), -(gu - \alpha\psi(u)))^T$. Hence the eigenvalues are real and coincide with those of Euler system $u - c, u, u + c$ (where c denotes the Eulerian sound speed), only u is now a double eigenvalue. Now, because of the special form of (5.5), we may take as eigenvectors $\mathbf{R}_i(\mathbf{U}) = (\mathbf{r}_i(\mathbf{u}), 0)$ where we have noted by $\mathbf{r}_i$ the eigenvectors for Euler system and, if we want a basis of $\mathbb{R}^4$, we need to find a fourth eigenvector which is associated to u of the form $\mathbf{R} = (\mathbf{r}, 1)$, and a simple computation shows that this is possible only if $\psi(u) = u\varphi(u)$ or $u = 0$. □

Lemma 3. *The quantity $w(\mathbf{U}) = p - q(g - \alpha\varphi(u))$ is a Riemann invariant associated to u, i.e. $\nabla_{\mathbf{U}} w.\mathbf{R} = 0$ for any eigenvector $\mathbf{R}$ associated to u. The potential q is a Riemann invariant for the 1- and 4-characteristic fields.*

Proof. For what concerns the potential q, the result is straightforward because the 1- and 4-eigenvectors are of the form $\mathbf{R}_i = (\mathbf{r}_i, 0)$.

For the characteristic field u, we have also such an eigenvector say $\mathbf{R}^E = (\mathbf{r}^E, 0)$, and since $\nabla_{\mathbf{U}} p = (\nabla_{\mathbf{u}} p, 0)$, we have $\nabla_{\mathbf{U}} p.\mathbf{R}^E = 0$ because p is a Riemann invariant for Euler system. Then we compute $\nabla_{\mathbf{U}}(q(g - \alpha\varphi(u))) = (\alpha\varphi'(u)uq/\varrho, -\alpha\varphi'(u)q/\varrho, 0, g - \alpha\varphi(u))$ and if $\mathbf{r}^E = (r_1^E, r_2^E, r_3^E)^T$, it yields

$$\nabla_{\mathbf{U}} w.\mathbf{R}^E = \nabla_{\mathbf{U}}(q(g - \alpha\varphi(u))).\mathbf{R}^E = -\alpha\varphi'(u)q(ur_1^E - r_2^E)/\varrho.$$

But any eigenvector of $\mathbf{A}(\mathbf{U})$ associated to u satisfies $r_2 = ur_1$ as results by identifying the first component of each side of the equality $\mathbf{AR} = u\mathbf{R}$, from which we deduce that $\nabla_{\mathbf{U}} w.\mathbf{R}^E = 0$.

Now, for the other eigenvector $\mathbf{R} = (\mathbf{r}, 1)$, writing that $\mathbf{R}$ is an eigenvector of $\mathbf{A}$, and taking the second component of both sides of the equality $\mathbf{AR} = u\mathbf{R}$, expliciting the components of $\mathbf{A}$ gives a relation

$$\nabla_{\mathbf{u}}(p + \varrho u^2).\mathbf{r} - (g - \alpha\varphi(u)) = ur_2$$

if $\mathbf{r} = (r_1, r_2, r_3)^T$. First, we have $\nabla_{\mathbf{u}}(\varrho u^2) = (-u^2, 2u)^T$ and thus $\nabla_{\mathbf{u}}(\varrho u^2).\mathbf{r} = -u^2 r_1 + 2ur_2$ which gives

$$\nabla_{\mathbf{U}} p.\mathbf{R} = \nabla_{\mathbf{u}} p.\mathbf{r} = -ur_2 + u^2 r_1 + g - \alpha\varphi(u).$$

Since again $r_2 = ur_1$, we deduce that $\nabla_{\mathbf{U}} p.\mathbf{R} = g - \alpha\varphi(u)$. Then we compute $\nabla_{\mathbf{U}}((g - \alpha\varphi(u))q) = (\alpha\varphi'(u)uq/\varrho, -\alpha\varphi'(u)q/\varrho, 0, g - \alpha\varphi(u))$ and

$$\nabla_{\mathbf{U}}(q(g - \alpha\varphi(u))).\mathbf{R} = \alpha\varphi'(u)(ur_1 - r_2)q/\varrho + g - \alpha\varphi(u) = g - \alpha\varphi(u),$$

which yields $\nabla_{\mathbf{U}} w.\mathbf{R} = 0$. □

One consequence is that when q is discontinuous, which may happen only across a contact discontinuity of speed u, the non-conservative product $(g - \alpha\varphi(u))\partial_x q$ is well defined; in that case the pressure p is no longer constant, as it is for the usual Euler system without source, and $[p] = (g - \alpha\varphi(u))[q]$.

Note now that if η is the mathematical specific entropy for Euler system, smooth solutions of (5.3) satisfy equality (2.4)

$$-T(\partial_t \varrho\eta + \partial_x(\varrho\eta u)) = \alpha\varrho(u\varphi(u) - \psi(u)),$$

and if (5.4) holds, (2.4) becomes a conservation law

$$\partial_t(\varrho\eta) + \partial_x(\varrho\eta u) = 0. \tag{5.6}$$

Thus, if (5.4) holds, the system (5.3) is hyperbolic, smooth solutions of system (5.1) satisfy

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \partial_t(\varrho\eta) + \partial_x(\varrho\eta u) = 0, \end{cases} \tag{5.7}$$

(with $p = \tilde{p}(\tau, \eta), \tau = 1/\varrho$) and smooth solutions of (5.3) satisfy

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) - (g - \alpha\varphi(u))\partial_x q = 0 \\ \partial_t(\varrho\eta) + \partial_x(\varrho\eta u) = 0 \\ \partial_t(\varrho q) + \partial_x(\varrho u q) = 0, \end{cases} \tag{5.8}$$

(with again $p = \tilde{p}(\tau, \eta), \tau = 1/\varrho$).

We can prove:

Lemma 4. *Let $(\varrho, u, \eta, q)(x,t)$ be a smooth solution of (5.3) associated to a given initial data $(\varrho_0, u_0, \eta_0, q_0)(x)$ such that q_0 satisfies $\frac{dq_0}{dx}(x) = \varrho_0(x)$. Then $q(x,t)$ satisfies $\partial_x q(x,t) = \varrho(x,t)$ and $(\varrho, u, \eta)(x,t)$ is a smooth solution of system (5.7) associated to $(\varrho_0, u_0, \eta_0)(x)$.*

Proof. Let $(\varrho, u, \eta)(x,t)$ be the smooth solution of (5.7) associated to the data $(\varrho_0, u_0, \eta_0)(x)$ (this solution exists at least for t small enough). Define, associated to this solution, the function $\overline{q}(x,t)$ by $\partial_x \overline{q} = \varrho, \partial_t \overline{q} = -\varrho u$. Then $(\varrho, u, \eta, \overline{q})(x,t)$ is the smooth solution of (5.8) associated to $(\varrho_0, u_0, \eta_0, q_0)(x)$, hence $q = \overline{q}$. □

Hence we can use the formulation with potential which is equivalent for smooth solutions. Now for discontinuous solutions, the Rankine-Hugoniot relations for (5.3) give that when a discontinuity propagates with speed σ:

- either $\sigma \neq u$, we have a shock and then $[q] = 0$, q is continuous and we have the same jump relations as for Euler (5.1),
- or $\sigma = u$, we have a contact discontinuity, $[p] = (g - \alpha\varphi(u)[q]$ (equivalently, $w = p - (g - \alpha\varphi(u))q$ is a Riemann invariant). If $[q] = 0$, we recover $[p] = 0$ as for Euler.

Thus, the formulation (5.3) introduces possible discontinuities of q propagating with velocity u but no new discontinuities for (5.1) and shocks propagate at the right speed. Given two constant states $(\varrho_o, \varrho_o u_o, \varrho_o e_o, q_o), o = L, R$, close enough, we can

solve the Riemann problem for (5.3) following the same steps used to solve the Riemann problem for Euler.

5.2. *Equilibrium states, stationary solutions and asymptotic behavior*

Smooth solutions for (5.3) with $u = 0$ ('equilibrium' solutions) satisfy

$$\begin{cases} \partial_t \varrho = 0, \\ \partial_x p = g \partial_x q \\ \partial_t \varepsilon = 0 \\ \partial_t q = 0, \end{cases} \tag{5.9}$$

thus also $\partial_t p = 0$, $\varrho = \varrho(x), p = p(x)$ and we have stationary solutions for what concerns density and pressure, and $p - gq$ is constant. If we take $\frac{dq}{dx} = \varrho(x)$, then $\frac{dp}{dx} = g\varrho(x)$ as in (2.9). Note that in Ref. 9, for $\alpha = 0$, the authors are interested by preserving at the numerical level, a particular equilibrium, an atmospheric column at rest, thus satisfying the same equations, $u = 0$, only $p = p(x)$ is given from tabulated data, and ϱ solves $\frac{dp}{dx} = g\varrho(x)$.

Smooth stationary solutions for (5.3) satisfy

$$\begin{cases} \frac{d}{dx}(\varrho u) = 0, \\ \frac{d}{dx}(\varrho u^2 + p) - (g - \alpha\varphi(u))\frac{d}{dx}q = 0, \\ \frac{d}{dx}((\varrho e + p)u) - (gu - \alpha\psi(u))\frac{d}{dx}q = 0, \\ \frac{d}{dx}(\varrho u q) = 0. \end{cases} \tag{5.10}$$

Note that because of (5.2), we cannot expect stationary solution with $u(x) = \overline{u}$ constant and $\overline{u} \neq 0$ if the density does not vanish since $\partial_t q = -\varrho u$.

If $\overline{u} = 0$, we obtain the 'equilibrium' solutions with $u = 0$ described above.

Discontinuous stationary solutions for (5.3) satisfy Rankine-Hugoniot jump conditions with $\sigma = 0$. If q may be discontinuous, then $\sigma = u = 0$, we have a stationary contact discontinuity, and $[p] = g[q]$, $p - gq$ is continuous (it is again an equilibrium). If q is continuous, $[q] = 0$, the stationary discontinuity is a sonic ($u = \pm c$) discontinuity for Euler which we do not consider here.

Now, performing the same scaling as previously (see (3.43))

$$t = \beta s, \quad v = \beta u, \quad \beta = \alpha^{\frac{1}{\chi+1}},$$

the equations (5.2) for q become

$$\begin{cases} \partial_x q = \varrho \\ \partial_s q = -\varrho\beta u = -\varrho v \end{cases} \tag{5.11}$$

and the equation for q is invariant $\partial_s(\varrho q) + \partial_x(\varrho v q) = 0$. Thus the asymptotic behavior of the original system (5.7) is preserved. Formally, after scaling, as $\beta \to \infty$,

at the order 0 in $\frac{1}{\beta}$, the solutions of (5.3) tend to those of

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \partial_x p - (g - \alpha\varphi(v))\partial_x q = 0, \\ \partial_s(\varrho\varepsilon) + \partial_x((\varrho\varepsilon + p)v) - (gv - \alpha\psi(v))\partial_x q = 0, \\ \partial_s(\varrho q) + \partial_x(\varrho v q) = 0, \end{cases} \tag{5.12}$$

and again, if the data are 'well prepared' (i.e. satisfy $q_0'(x) = \varrho_0(x)$), $(\varrho, \varepsilon, v)$ is solution of (2.22).

5.3. *Numerical relaxation*

The idea developed in Ref. 9 is to use an exact or approximate Riemann solver associated to the formulation with potential (5.3) to define a well balanced scheme. In order to solve (5.1) written in the form (5.3), our approximate solver will involve an exact Riemann solver for a relaxation system with LD fields built from (5.8), which, at least from a heuristic point of view, is naturally asymptotic preserving.

Indeed, following previous work,[10,11] we introduce a larger 5×5 system with a relaxation term in the right-hand side depending on a 'relaxation parameter' ν, which is meant to become arbitrarily large (so that the 'relaxation time' $\frac{1}{\nu}$ goes to 0)

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) - (g - \alpha\varphi(u))\partial_x q & = 0, \\ \partial_t \varrho\eta + \partial_x(\varrho\eta u) + \frac{\alpha}{T}(u\varphi(u) - \psi(u))\partial_x q & = 0, \\ \partial_t(\varrho\mathcal{T}) + \partial_x(\varrho\mathcal{T} u) & = \nu\varrho(\tau - \mathcal{T}), \\ \partial_t(\varrho q) + \partial_x(\varrho u q) & = 0, \end{cases} \tag{5.13}$$

where T is the temperature (the third equation comes from (2.4)). This new system (5.13) needs a closure relation for Π which, following Ref. 11, we take in the form

$$\Pi = \tilde{\Pi}(\tau, \eta, \mathcal{T}) \equiv \tilde{p}(\mathcal{T}, \eta) + \overline{C}^2(\mathcal{T} - \tau). \tag{5.14}$$

Here $\overline{C}$ is a positive constant which plays the role of a frozen Lagrangian sound speed and is required to bound the exact sound speed (previouly noted $C^{\mathcal{L}}$, $(C^{\mathcal{L}})^2 = -\partial_\tau \tilde{p}(\tau, \eta)$) for Euler system: this is the Whitham (or subcharacteristic) condition

$$\overline{C}^2 > -\partial_\tau \tilde{p}(\tau, \eta), \tag{5.15}$$

for all the values τ, η under consideration. Formally, as the relaxation parameter $\nu \to \infty$, $\mathcal{T} - \tau \to 0$, and $\tilde{\Pi}(\tau, \eta, \mathcal{T}) \to \tilde{p}(\tau, \eta) = p$ and we indeed recover Euler system with friction at *equilibrium* with a formulation involving a potential, where the energy equation has been replaced by the entropy one (2.4). Here the term *equilibrium* is relative to the relaxation procedure: as $\nu \to \infty$, the solution 'relaxes' to an *equilibrium*. Note that, without an additional step, this *equilibrium* would not be consistent with the weak form of (5.12) since the weak solutions we consider are those of system (5.13) which preserves the entropy $\varrho\eta$ and makes the total energy

play the role of a convex entropy. The step required to restore the conservation of total energy is simple and will be detailed in section 5.3.3 below.

5.3.1. *Study of the relaxation system*

If we write (5.13) in the condensed form $\partial_t \mathbf{U} + F(\partial_x \mathbf{U}, \mathbf{U}) = \nu \mathbf{S}(\mathbf{U})$, the convective part of the system ($\nu = 0$) has explicit Riemann solutions, and the scheme results from a splitting between a convection step (solving (5.13) for $\nu = 0$, thus $\partial_t \mathbf{U} + F(\partial_x \mathbf{U}, \mathbf{U}) = 0$ with Godunov's scheme) and the treatment of the source in a relaxation step (thus $\partial_t \mathbf{U} = \nu \mathbf{S}(\mathbf{U})$ for $\nu \to \infty$), which since the relaxation is instantaneous, can be considered as a projection on the *equilibrium* manifold, $\mathbf{S}(\mathbf{U}) = 0$ (here, the set of states $\mathbf{U}$ such that $\mathcal{T} = \tau$). We will see that, since the Riemann problem has explicit solutions, the numerical fluxes can be computed and the scheme, when restricting to the variables ϱ, u, e, gives a consistent scheme for (5.1).

For the convective part of the augmented system, i.e. system (5.13) when $\nu = 0$:

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) - (g - \alpha\varphi(u))\partial_x q & = 0 \\ \partial_t \varrho\eta + \partial_x(\varrho\eta u) + \frac{\alpha}{T}(u\varphi(u) - \psi(u))\partial_x q & = 0 \\ \partial_t(\varrho\mathcal{T}) + \partial_x(\varrho\mathcal{T} u) & = 0 \\ \partial_t \varrho q + \partial_x(\varrho u q) & = 0, \end{cases} \tag{5.16}$$

let us set $\mathbf{U} = (\varrho, \varrho u, \varrho\eta, \varrho\mathcal{T}, q)^T$, $\mathcal{J} = \varrho\mathcal{T}$ and $S = \varrho\eta$, the state $\mathbf{U}$ belongs to the set $\Omega^{\eta,q} = \{(\varrho, \varrho u, S = \varrho\eta, \mathcal{J} = \varrho\mathcal{T}, q); \varrho > 0, u \in \mathbb{R}, \eta > 0, \mathcal{T} > 0, q \in \mathbb{R}\}$. We will say that a state $\mathbf{U}$ is at *equilibrium* if $\mathcal{T} = \tau$ so that $\Pi = \tilde{p}(\tau, \eta)$. Defining, the mapping $\Pi(\varrho, \varrho\eta, \varrho\mathcal{T}) \equiv \tilde{\Pi}(\tau, \eta, \mathcal{T})$, the system (5.16) is in quasilinear form $\partial_t \mathbf{U} + \mathbf{A}^{q,R}(\mathbf{U})\partial_x \mathbf{U}$ with matrix given by

$$\mathbf{A}^{q,R}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -u^2 + \Pi_\varrho & 2u & \Pi_S & \Pi_\mathcal{J} & A_{q,2} \\ -\eta u & \eta & u & 0 & A_{q,3} \\ -\mathcal{T} u & \mathcal{T} & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix} \tag{5.17}$$

where $A_{q,2} = -(g - \alpha\varphi(u))$, $A_{q,3} = \frac{\alpha}{T}(u\varphi(u) - \psi(u))$. We can prove:

Lemma 5. *System (5.16) is hyperbolic iff condition (5.4) holds. The eigenvalues of the matrix (5.17) are* $\lambda_1(\mathbf{U}) = u - \overline{C}\tau < \lambda_2 = \lambda_3 = \lambda_4(\mathbf{U}) = u < \lambda_5(\mathbf{U}) = u + \overline{C}\tau$. *The corresponding eigenvectors* $\mathbf{R}_i = \mathbf{R}_i(\mathbf{U})$ *may be taken as*

$$\mathbf{R}_1 = \begin{pmatrix} 1 \\ u - \overline{C}\tau \\ \eta \\ \mathcal{T} \\ 0 \end{pmatrix}, \mathbf{R}_2 = \begin{pmatrix} 1 \\ u \\ -\Pi_\varrho/\Pi_S \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}_3 = \begin{pmatrix} 1 \\ u \\ 0 \\ -\Pi_\varrho/\Pi_\mathcal{J} \\ 0 \end{pmatrix},$$

$$\mathbf{R}_4 = \begin{pmatrix} 0 \\ 0 \\ -A_{q,2}/\Pi_S \\ 0 \\ 1 \end{pmatrix}, \mathbf{R}_5 = \begin{pmatrix} 1 \\ u + \overline{C}\tau \\ \eta \\ \mathcal{T} \\ 0 \end{pmatrix}.$$

Moreover, the five characteristic fields are linearly degenerate.

In the sequel, we will assume that condition (5.4) holds. We also have

Lemma 6. *Assume that $\overline{C}$ satisfies (5.15). Let $\mathbf{U}_L$ be a given state, then the wave curves $\mathcal{C}_i(\mathbf{U}_L), i = 1, 4$, and wave set $\mathcal{C}_{2,3,4}$ can be characterized as follows:*

- *The curve $\mathcal{C}_1(\mathbf{U}_L)$ is given by $\mathcal{C}_1(\mathbf{U}_L) = \{\mathbf{U} \in \Omega^{\eta,q}; u = u_L - \frac{1}{\overline{C}}(\Pi - \Pi_L), \eta = \eta_L, \mathcal{T} = \mathcal{T}_L, q = q_L\}$.*
- *The set $\mathcal{C}_{2,3,4}(\mathbf{U}_L)$ is given by $\mathcal{C}_{2,3,4}(\mathbf{U}_L) = \{\mathbf{U} \in \Omega^{\eta,q}; u = u_L, W = W_L\}$, where $W(\mathbf{U}) = \Pi - q(g - \alpha\varphi(u))$.*
- *The curve $\mathcal{C}_5(\mathbf{U}_L)$ is given by $\mathcal{C}_3(\mathbf{U}_L) = \{\mathbf{U} \in \Omega^{\eta,q}; u = u_L + \frac{1}{\overline{C}}(\Pi - \Pi_L), \eta = \eta_L, \mathcal{T} = \mathcal{T}_L, q = q_L\}$.*

In (5.15), the supremum is taken over all possible values of (τ, η) occuring in the solution of a Riemann problem.

Proof. The proof for $\mathcal{C}_1, \mathcal{C}_5$ is easy. Also, we note that Whitham's condition (5.15) yields that the mapping $y \mapsto \tilde{p}(y, \eta) + \overline{C}^2 y$ is invertible. Now, since u is a triple eigenvalue, we can find only two Riemann invariants with independent gradients, which are again u since the field is LD and following the lines in the proof of Lemma 3, we find that the other is $W = \Pi - (g - \alpha\varphi(u))q$. □

The solution of the Riemann problem follows easily from the explicit knowledge of the Riemann invariants. To simplify the notations, we shall denote as previously, for any quantity, say b, $\Delta b = b_R - b_L$.

Proposition 5.1.
Given two constant states $\mathbf{U}_L$, $\mathbf{U}_R$, the solution $W_q(x/t; \mathbf{U}_L, \mathbf{U}_R)$ of the Riemann problem for (5.16), consists of three contact discontinuities, each propagating with a characteristic speed (resp. $u_L - \overline{C}\tau_L, u^*, u_R + \overline{C}\tau_R$), separating $\mathbf{U}_L$, two intermediate states $\mathbf{U}_L^*, \mathbf{U}_R^*$ and $\mathbf{U}_R$. The states $\mathbf{U}_L^*, \mathbf{U}_R^*$ are respectively characterized by $(u^*, \Pi_L^*; \mathcal{T}_L, \tau_L^*, q_L)$ and $(u^*, \Pi_R^*; \mathcal{T}_R, \tau_R^*, q_R)$ with u^* solving

$$u^* + \frac{\alpha}{2\overline{C}}\varphi(u^*)\Delta q = \frac{u_L + u_R}{2} - \frac{1}{2\overline{C}}(\Delta\Pi - g\Delta q), \tag{5.18}$$

and the other quantities satisfying

$$\begin{aligned}
\Pi_L^* &= \frac{\Pi_L + \Pi_R}{2} - \frac{\overline{C}}{2}\Delta u - \frac{1}{2}(g - \alpha\varphi(u^*))\Delta q \\
\Pi_R^* &= \frac{\Pi_L + \Pi_R}{2} - \frac{\overline{C}}{2}\Delta u + \frac{1}{2}(g - \alpha\varphi(u^*))\Delta q \\
\tau_L^* &= \tau_L + \frac{1}{\overline{C}}(u^* - u_L), \\
\tau_R^* &= \tau_R + \frac{1}{\overline{C}}(u_R - u^*).
\end{aligned} \tag{5.19}$$

Proof. First, using the $2, 3, 4$-Riemann-invariants, we have $u_L^* = u_R^* \equiv u^*$ and

$$\Pi_R^* - q_R(g - \alpha\varphi(u^*)) = \Pi_L^* - q_L(g - \alpha\varphi(u^*))$$

hence

$$\Delta\Pi^* \equiv \Pi_R^* - \Pi_L^* = (g - \alpha\varphi(u^*))\Delta q. \tag{5.20}$$

Then, projecting $\mathcal{C}_1(\mathbf{U}_L)$ and $\mathcal{C}_5(\mathbf{U}_R)$ given by Lemma 6, on the $(u, \Pi)-$plane gives

$$u_L^* = u_L - \frac{1}{\overline{C}}(\Pi_L^* - \Pi_L),\ u_R^* = u_R + \frac{1}{\overline{C}}(\Pi_R^* - \Pi_R)$$

both expressions beeing equal to u^* so that

$$u^* = \frac{u_L + u_R}{2} - \frac{1}{2\overline{C}}\Delta\Pi + \frac{1}{2\overline{C}}\Delta\Pi^*$$

and thus with (5.20)

$$u^* = \frac{u_L + u_R}{2} - \frac{1}{2\overline{C}}\Delta\Pi + \frac{1}{2\overline{C}}(g\Delta q - \alpha\varphi(u^*)\Delta q)$$

which means that u^* is the solution of

$$u^* + \frac{\alpha}{2\overline{C}}\varphi(u^*)\Delta q = \frac{u_L + u_R}{2} - \frac{1}{2\overline{C}}\Delta\Pi + \frac{1}{2\overline{C}}g\Delta q.$$

Also

$$\Pi_R^* + \Pi_L^* = \Pi_R + \Pi_L - \overline{C}\Delta u,$$

again with (5.20), it gives

$$\Pi_R^* = \frac{\Pi_L + \Pi_R}{2} - \frac{\overline{C}}{2}\Delta u + \frac{1}{2}(g - \alpha\varphi(u^*))\Delta q$$

$$\Pi_L^* = \frac{\Pi_L + \Pi_R}{2} - \frac{\overline{C}}{2}\Delta u - \frac{1}{2}(g - \alpha\varphi(u^*))\Delta q.$$

We may define

$$\Pi^* = \frac{\Pi_L^* + \Pi_R^*}{2} = \frac{\Pi_L + \Pi_R}{2} - \frac{\overline{C}}{2}\Delta u,$$

and we check that when g and α vanish, $\Pi_R^* = \Pi_L^* = \Pi^*$. □

Let us introduce the 'entropy' (in fact the energy) for system (5.13)

$$\begin{aligned}\Sigma(\tau,u,\eta,\mathcal{T}) &= \tilde{\varepsilon}(\mathcal{T},\eta)+\frac{u^2}{2}+\frac{\Pi^2(\tau,\eta,\mathcal{T})-\tilde{p}^2(\mathcal{T},\eta)}{2\overline{C}^2}\\ &= \tilde{\varepsilon}(\mathcal{T},\eta)+\frac{u^2}{2}+\frac{\Pi(\tau,\eta,\mathcal{T})+\tilde{p}(\mathcal{T},\eta)}{2}(\mathcal{T}-\tau)\end{aligned} \tag{5.21}$$

where $\tilde{\varepsilon}(\tau,\eta)=\varepsilon(\varrho,\eta)$ is known to satisfy $\partial_\tau\tilde{\varepsilon}=-\tilde{p}$.

Since system (5.16) is linearly degenerate, we do not need convexity to select its admissible solutions. However, in order to justify the relaxation procedure, we prove an inequality when we take into account the relaxation term of (5.13). Assuming (5.4), we can prove

Proposition 5.2. *Let Σ be defined by (5.21). Then, for* equilibrium *states we have*

$$\Sigma(\tau,u,\eta,\tau)=e.$$

Smooth solutions of (5.16) satisfy

$$\partial_t\varrho\Sigma+\partial_x((\varrho\Sigma+\Pi)u)-(gu-\alpha u\varphi(u))\partial_x q=-\nu\varrho(\partial_\tau\tilde{p}(\mathcal{T},\eta)+\overline{C}^2)(\tau-\mathcal{T})^2, \tag{5.22}$$

which is negative if condition (5.15) holds. For a given $\mathbf{U}=(\varrho,\varrho u,\varrho\eta,\varrho\mathcal{T},q)$, let us note $\mathbf{U}^{eq}=(\varrho,\varrho u,\varrho\eta,1,q)$. We have the following Gibbs principle:

$$\varrho e=(\varrho\Sigma)(\mathbf{U}^{eq})=\min_{\mathcal{T}\in K}(\varrho\Sigma)(\mathbf{U}). \tag{5.23}$$

In (5.23), K is a compact set such that it contains all values of $\mathcal{T}$ under consideration.

Proof. Assuming that $\psi(u)=u\varphi(u)$, we have for a smooth solution of (5.13)

$$\varrho\partial_t\Pi(\tau,\eta,\mathcal{T})+\varrho u\partial_x\Pi(\tau,\eta,\mathcal{T})+\overline{C}^2\partial_x u=-\nu\varrho(\partial_\tau(\tilde{p}(\mathcal{T},\eta)+\overline{C}^2)(\tau-\mathcal{T}).$$

Then from the second equation we get

$$\partial_t\frac{\varrho u^2}{2}+\partial_x u(\frac{\varrho u^2}{2}+\Pi)-\Pi\partial_x u-u(g-\alpha\varphi(u))\partial_x q=0$$

and combining the two yields

$$\partial_t(\frac{\varrho u^2}{2}+\frac{\varrho\Pi^2}{2\overline{C}^2})+\partial_x((\frac{\varrho u^2}{2}+\frac{\varrho\Pi^2}{2\overline{C}^2}+\Pi)u)-u(g-\alpha\varphi(u))\partial_x q$$

$$=\nu\varrho\frac{\Pi}{\overline{C}^2}(\partial_\tau\tilde{p}(\mathcal{T},\eta)+\overline{C}^2)(\tau-\mathcal{T}).$$

The remaining lines to get (5.22) follow easily.

For the proof of (5.23), we differentiate the function Σ with respect to $\mathcal{T}$ and get $\partial_{\mathcal{T}}\Sigma=(\partial_\tau\tilde{p}+\overline{C}^2)(\mathcal{T}-\tau)$ which vanishes only for $\mathcal{T}=\tau$ with a strictly positive second order derivative at $\mathcal{T}=\tau$ if again (5.15) is satisfied. □

Remark 5.2. By (5.22), smooth solutions of (5.16) satisfy

$$\partial_t(\varrho\Sigma) + \partial_x((\varrho\Sigma + \Pi)u) - (gu - \alpha u\varphi(u))\partial_x q = 0, \tag{5.24}$$

but it also holds for discontinuous solutions since all the fields are linearly degenerate, in particular the field associated to u across which q is discontinuous. We have seen in lemma 6 that $W = \Pi - q(g - \alpha\varphi(u))$ is also constant, hence the jump condition for (5.24) across such a contact discontinuity is also satisfied. □

Now, with this definition of Σ, we can check

Lemma 7. *If the Riemann data are at* equilibrium *($\mathcal{T}_L = \tau_L$, $\mathcal{T}_R = \tau_R$), the energy of the two intermediate states defined in Proposition 5.1 satisfies*

$$\begin{aligned}\Sigma_L^* &= \Sigma_L - \tfrac{1}{\overline{C}}(\Pi^* u^* - \Pi_L u_L) + \tfrac{1}{2\overline{C}} u^*(g - \alpha\varphi(u^*))\Delta q \\ \Sigma_R^* &= \Sigma_R + \tfrac{1}{\overline{C}}(\Pi^* u^* - \Pi_R u_R) + \tfrac{1}{2\overline{C}} u^*(g - \alpha\varphi(u^*))\Delta q,\end{aligned} \tag{5.25}$$

where $\Pi^* = \frac{1}{2}(\Pi_L^* + \Pi_R^*) = \frac{1}{2}(\Pi_L + \Pi_R) - \frac{\overline{C}}{2}\Delta u$.

Proof. Indeed, from (5.21) and lemma 6, we have across a 1-wave

$$[\Sigma] = \frac{1}{2\overline{C}^2}(\overline{C}^2[u^2] + [\Pi^2]) = \frac{1}{2\overline{C}^2}(\overline{C}^2[u](u^* + u_L) + [\Pi](\Pi_L^* + \Pi_L))$$

together with $[\Pi] + \overline{C}[u] = 0$, it yields $[\Sigma] = -\frac{1}{2\overline{C}}([\Pi](u^* + u_L) + [u](\Pi_L^* + \Pi_L))$ thus

$$[\Sigma] = -\frac{1}{\overline{C}}[\Pi u]$$

so that $\Pi u + \overline{C}\Sigma$ is also a 1-Riemann invariant. Similarly, we find that $\Pi u - \overline{C}\Sigma$ is a 3-Riemann invariant and the expressions (5.25) of $\Sigma_{L,R}^*$ follow from the expressions (5.19) of $\Pi_{L,R}^*$. Note that (5.25) corresponds precisely to formulas (3.34) where $\Sigma \equiv e$. □

5.3.2. *The global relaxation scheme*

Let us now define the resulting scheme which involves a fractional step method to advance the solution in time from t_n to $t_{n+1} = t_n + \Delta t$ with three steps: reconstruction, evolution, projection. Before giving the details, we describe the main lines of the relaxation part of the scheme, and for $n = 0$, since they are similar at all other time. Let $\mathbf{u}_0(x) = (\varrho_0, \varrho_0 u_0, \varrho_0 e_0)^T(x)$ be an initial datum for system (5.3):

(1) Define the extended initial datum $\mathbf{U}_0(x) = (\varrho_0, \varrho_0 u_0, \varrho_0\eta_0, \varrho_0\mathcal{T}_0, q_0)^T(x)$ for the relaxation system (5.13), where $\eta_0 = \eta(\mathbf{u}_0)$ and defining $\mathcal{T}_0 \equiv 1/\varrho_0$: $\mathbf{U}_0$ is at *equilibrium.*
(2) Solve the Cauchy problem (5.13), (5.14) with the initial data $\mathbf{U}_0$ for $t \in (0, \Delta t]$ we obtain $\mathbf{U}_1^-(x) = \mathbf{U}(x, \Delta t)$.

(3) Project $\mathbf{U}_1^- = (\varrho_1, \varrho_1 u_1, \varrho_1 \eta_1, \varrho_1 \mathcal{T}_1, q_1)^T$ on the *equilibrium* set of system (5.13) (instantaneous relaxation) to get $\mathbf{U}_1 = (\varrho_1, \varrho_1 u_1, \varrho_1 \eta_1, 1, q_1)$.
(4) Define $\mathbf{u}_1(x) = (\varrho_1, \varrho_1 u_1, \varrho_1 e_1)^T(x)$.

In fact step 4 is straightforward at the computational level but requires a careful analysis to justify the scheme for $\mathbf{u} = (\varrho, \varrho u, \varrho e)^T$ from the quantities $\mathbf{U} = (\varrho, \varrho u, \varrho \eta, \varrho \mathcal{T}, q)^T$. The required material will be given in the next section.

The 'potential' q needs only be defined for the full discretization and through Δq. As usual $\mathbf{u}_0(x)$ is first discretized

$$\mathbf{u}_j^0 = \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathbf{u}_0(x) dx, \tag{5.26}$$

and we define $\Delta q_{j+1/2} = \frac{\varrho_j + \varrho_{j+1}}{2} \Delta x$ (or some other consistent average of ϱ_j, ϱ_{j+1}, see (4.18)), which is of course coherent with (5.2). In steps 2 and 3 we will use a Godunov-type solver, which means that we solve exactly a juxtaposition of Riemann problems associated to the piecewise constant data $\mathbf{U}_\Delta(x, t_n)$ (function equal to $\mathbf{U}_j^n$ on C_j) and project the solution back on the grid, this results in a global relaxation solver (it is a simple solver in the sense given in section 3.1).

Let us summarize the resulting global relaxation method for approximating Euler system with friction (5.3).

Starting from an initial condition $\mathbf{u_0}(x)$ discretized by $\mathbf{u}_j^0 = (\varrho, \varrho u, \varrho e)_j^0$, $j \in \mathbb{Z}$, setting $\mu = \Delta t / \Delta x$,
- define $\mathbf{U}_j^n = ((\varrho, \varrho u, \varrho \eta)_j^n, 1)^T$ and $\Delta q_{j+1/2}$ the extended *equilibrium* state.
- solve the Riemann problems $\mathbf{W}_q(.; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$, $j \in \mathbb{Z}$, using the results of Proposition 5.1, we know explicitly the intermediate states,
- using Godunov's method define the update value $\mathbf{U}_j^{n+1-}$

$$\left\{ \begin{array}{l} \mathbf{U}_j^{n+1-} = \frac{1}{\Delta x} \Big(\int_0^{\Delta x/2} \mathbf{W}_q(\frac{x}{\Delta t}; \mathbf{U}_{j-1}^n, \mathbf{U}_j^n) dx \\ \qquad\qquad + \int_{-\Delta x/2}^0 \mathbf{W}_q(\frac{x}{\Delta t}; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n) dx \Big), \end{array} \right. \tag{5.27}$$

it can be written in the form

$$\mathbf{U}_j^{n+1-} = \mathbf{U}_j^n - \mu \left(\mathbf{G}_{j+1/2-}^n - \mathbf{G}_{j-1/2+}^n \right), \quad j \in \mathbb{Z}, n \geq 0, \tag{5.28}$$

where $\mathbf{G}_{j+1/2\pm}^n = (\mathcal{G}_{\varrho,j+1/2}^n, \mathcal{G}_{\varrho u,j+1/2\pm}^n, \mathcal{G}_{\varrho\eta,j+1/2}^n, \mathcal{G}_{\varrho\mathcal{T},j+1/2}^n, \mathcal{G}_{\varrho q,j+1/2}^n)^T$ has in fact only its second component which is non conservative.
- Keep the two first components for the two first components of $\mathbf{u}_j^{n+1}$. This results in a conservative discrete equation for ϱ_j^{n+1}, since there is no source term

$$\varrho_j^{n+1} = \varrho_j^n - \mu \left(\mathcal{G}_{\varrho,j+1/2}^n - \mathcal{G}_{\varrho,j-1/2}^n \right), \quad j \in \mathbb{Z}, n \geq 0, \tag{5.29}$$

and a formula for $(\varrho u)_j^{n+1}$

$$(\varrho u)_j^{n+1} = (\varrho u)_j^n - \mu \left(\mathcal{G}_{\varrho u,j+1/2-}^n - \mathcal{G}_{\varrho u,j-1/2+}^n \right), \quad j \in \mathbb{Z}, n \geq 0, \tag{5.30}$$

where the source term has been taken into account.

Here, we postpone the details of step 4 which follows Ref. 11 for defining the energy in the global solver so as to obtain a standard conservative scheme with an energy component of the flux, which will be written in the form (see (5.35) below)

$$E_j^{n+1} = E_j^n - \mu\left(\mathcal{G}_{\mathcal{E},j+1/2}^n - \mathcal{G}_{\mathcal{E},j-1/2}^n\right), \quad j \in \mathbb{Z}, n \geq 0.$$

It will be the object of next section 5.3.3. Once the above equation for the energy is given, the definition of the updated state $\mathbf{u}_j^{n+1}$ is complete.

Moreover, using (5.23), an entropy inequality can be established. Note that for $(\varrho\eta)$, the value obtained after the same steps described for ϱ and ϱu is not the value of the updated state, hence we note it with superscript $n+1-$ which gives

$$(\varrho\eta)_j^{n+1-} = (\varrho\eta)_j^n - \mu\big(\mathcal{G}_{\varrho\eta,j+1/2}^n - \mathcal{G}_{\varrho\eta,j-1/2}^n\big), \tag{5.31}$$

where $\mathcal{G}_{\varrho\eta,j+1/2}^n$ is the third component of Godunov's flux in (5.28). The value $(\varrho\eta)_j^{n+1} \equiv (\varrho\eta)(\mathbf{u}_j^{n+1})$ of the updated state will be such that we have:

Proposition 5.3. *The global relaxation solver satisfies a discrete entropy inequality*

$$(\varrho\eta)_j^{n+1} \leq (\varrho\eta)_j^n - \mu\big(\mathcal{G}_{\varrho\eta,j+1/2}^n - \mathcal{G}_{\varrho\eta,j-1/2}^n\big), \tag{5.32}$$

where the discrete entropy flux $\mathcal{G}_{\varrho\eta,j+1/2}^n$ defined by the third component of Godunov's flux (5.28) is consistent with the exact entropy flux.

The resulting scheme coincides with the one constructed in section 4.3 provided we set $\Delta q = \Delta m$, where recall that Δm has been defined by $\frac{\varrho_L+\varrho_R}{2}\Delta x$ (or some other consistent average of ϱ_L, ϱ_R, see (4.18)), which is of course coherent with (5.2), and if we choose $C = \overline{C}$. In this approach, the value of $\tilde{u}$ in (4.26) is directly given by u^*.

5.3.3. *Definition of the numerical energy flux and discrete entropy inequalities*

Let $\varrho\Sigma$ be defined by (5.21), we note $\mathcal{E} = \varrho\Sigma$ and $E = \varrho e$ the total energies and

$$\mathcal{G}(\mathbf{U}) = (\varrho\Sigma + \Pi)u = (\mathcal{E} + \Pi)u. \tag{5.33}$$

Note $\mathbf{U}(x, t_{n+1}-)$ the value at t_{n+1} of the solution obtained after the evolution step 2 and $\mathcal{E}(\mathbf{U}(x, t_{n+1}-))$ its energy. Now $\mathcal{E}$ is not a component of $\mathbf{U}$. However, because we use the exact Riemann solver at step 2 and our system is linearly degenerate, we get by integrating the energy equation (5.24) the analogue of (5.27) (which concerns the components of $\mathbf{U}$) for this energy

$$\frac{1}{\Delta x}\int_{C_j} \mathcal{E}(\mathbf{U}(x, t_{n+1}-))dx = \mathcal{E}_j^n - \mu\big(\mathcal{G}_{\mathcal{E},j+1/2-}^n - \mathcal{G}_{\mathcal{E},j-1/2+}^n\big). \tag{5.34}$$

The state $\mathbf{U}_j^n$ is at *equilibrium*, hence we have $\mathcal{E}_j^n = \varrho e(\mathbf{u}_j^n) = E_j^n$. Define the updated value E_j^{n+1} by the right-hand side of (5.34)

$$E_j^{n+1} = E_j^n - \mu\big(\mathcal{G}_{\mathcal{E},j+1/2-}^n - \mathcal{G}_{\mathcal{E},j-1/2+}^n\big). \tag{5.35}$$

Our scheme is now well defined by (5.29), (5.30) and (5.35), and it does coincide with that of section 4.3.

We want moreover to prove some discrete entropy inequalities. $\mathcal{E}$ is not convex and we cannot directly use Jensen's inequality, we must use a refined argument. For the projection step, assume for a while that the projection on the *equilibrium* manifold is done pointwise, i.e., define $\mathbf{U}(x, t_{n+1}^-)$ by

$$\begin{cases} \varrho(x, t_{n+1}^-) = \varrho(x, t_{n+1}-), \\ \varrho u(x, t_{n+1}^-) = \varrho u(x, t_{n+1}-), \\ \varrho\eta(x, t_{n+1}^-) = \varrho\eta(x, t_{n+1}-), \\ \mathcal{T}(x, t_{n+1}^-) = \tau(x, t_{n+1}-). \end{cases} \tag{5.36}$$

We use the minimization principle (5.23) of Proposition 5.2 which says that the maximal dissipation of entropy is attained for *equilibrium* states (K is a compact set such that it contains all possible values of τ obtained at the first step)

$$\mathcal{E}(\mathbf{U}^{eq}) = \min_{\mathcal{T}\in K} \mathcal{E}(\mathbf{U}).$$

Then, since for $\mathbf{U} = \mathbf{U}(x, t_{n+1}-)$, we have $\mathbf{U}^{eq} = \mathbf{U}(x, t_{n+1}^-)$, on the one side we deduce from (5.34), (5.35)

$$\frac{1}{\Delta x}\int_{C_j} \mathcal{E}(\mathbf{U}(x, t_{n+1}^-))dx \le \frac{1}{\Delta x}\int_{C_j} \mathcal{E}(\mathbf{U}(x, t_{n+1}-))dx = E_j^{n+1}, \tag{5.37}$$

on the other side we have

$$\mathcal{E}(\mathbf{U}(x, t_{n+1}^-)) = E(x, t_{n+1}^-). \tag{5.38}$$

Since the three first components $(\varrho, \varrho u, \varrho\eta)$ of $\mathbf{U}(x, t_{n+1}-)$ are not changed during the projection step, whether pointwise (5.36) or as initially defined, $\varrho_j^{n+1} = \frac{1}{\Delta x}\int_{I_j} \varrho(x, t_{n+1}^-)dx$ is given by the scheme (5.29) and $(\varrho u)_j^{n+1}$ by (5.30). For $\varrho\eta$, since it should change, we have noted it with superscript $n+1-$

$$\varrho\eta_j^{n+1-} = \frac{1}{\Delta x}\int_{C_j} \varrho\eta(x, t_{n+1}^-)dx$$

and it is given by (5.31)

$$\varrho\eta_j^{n+1-} = \varrho\eta_j^n - \mu\big(\mathcal{G}^n_{\varrho\eta,j+1/2} - \mathcal{G}^n_{\varrho\eta,j-1/2}\big).$$

Since the function $(\varrho, \varrho u, \varrho\eta) \to E$ is convex, we can apply Jensen's inequality to the expression $E(x, t_{n+1}^-) = E(\varrho, \varrho u, \varrho\eta)(x, t_{n+1}^-)$ and get, together with (5.37)(5.38)

$$E_j^{n+1-} \equiv E(\varrho_j^{n+1}, (\varrho u)_j^{n+1}, (\varrho\eta)_j^{n+1-}) \le \frac{1}{\Delta x}\int_{C_j} E(x, t_{n+1}^-)dx \le E_j^{n+1}.$$

Finally we use the fact that $\tilde{\varepsilon}$ satisfies $\partial_\eta\tilde{\varepsilon}(\tau, \eta) = -T < 0$, hence the mathematical entropy η is a decreasing function of the internal energy. Noting $S(\varrho, u, E) = \varrho\eta(\tau, \varepsilon)$, where $E = \varrho e = \varrho(\varepsilon + u^2/2)$, we get that $\partial_E S < 0$ and S is decreasing wrt. the third variable. Then define

$$(\varrho\eta)_j^{n+1} = S(\varrho_j^{n+1}, u_j^{n+1}, E_j^{n+1})$$

which is the definition of the entropy of $\mathbf{u}_j^{n+1}$. Hence

$$(\varrho\eta)_j^{n+1} = S(\varrho_j^{n+1}, u_j^{n+1}, E_j^{n+1}) \leq S(\varrho_j^{n+1}, u_j^{n+1}, E_j^{n+1-}) = (\varrho\eta)_j^{n+1-}$$

and together with (5.31), it yields

$$(\varrho\eta)_j^{n+1} \leq (\varrho\eta)_j^n - \mu\big(\mathcal{G}^n_{\varrho\eta,j+1/2} - \mathcal{G}^n_{\varrho\eta,j-1/2}\big).$$

We have thus proved the result of proposition 5.3.

5.3.4. *Properties*

The resulting scheme is well balanced in the sense that it preserves any equilibrium (2.9) for (5.3) which is discretized in a natural way.

Proposition 5.4. *Let $\boldsymbol{u}_0(x)$ be an equilibrium data for Euler system (5.3), i.e. satisfying $u_0 = 0, \partial_x p_0 = \varrho_0 g$. Assume that $\boldsymbol{u}_i^0$ is defined by (5.26) and define q^0 with $\Delta q = \frac{1}{g}\Delta p$. Then $\forall n > 0$, $\boldsymbol{u}_i^n = \boldsymbol{u}_i^0$.*

Proof. Let $\mathbf{u}_0(x)$ be a (non constant) equilibrium data for Euler, then the 'potential' satisfies $d_x p_0 = g d_x q_0$. This data is discretized for the scheme by piecewise constant functions, thus $u_i^0 = 0, \forall i \in \mathbb{Z}$, and we have $\frac{\Delta p}{\Delta x} = \varrho g + \mathcal{O}(\Delta x)$ (we might also start directly from a discrete equilibrium data satisfying $u_i = 0$ and $\frac{\Delta p_{i+1/2}}{\Delta x} = \frac{\varrho_i + \varrho_{i+1}}{2} g$ or any consistent discretization). We define $\mathcal{T}_i^0 = \tau_i^0$, thus $\Pi_i^0 = p_i^0$, and moreover Δq is defined such that $\Delta p = g\Delta q$.

Then for all local Riemann problems involved in Godunov's scheme, noting for simplicity by $\mathbf{U}_L, \mathbf{U}_R$ the states $\mathbf{U}_i^0, \mathbf{U}_{i+1}^0$ on each side of an interface $x_{i+1/2}$, from (5.18)(5.19) we get $u^* = 0$, $\tau^*_{L,R} = \tau_{L,R}$

$$\Pi^*_L = \frac{\Delta\Pi}{2} + \Pi_L - \frac{1}{2} g\Delta q = \Pi_L = p_L,$$

similarly $\Pi^*_R = \Pi_R = p_R$ the solution of the Riemann problem is indeed stationary. This is valid for any cell i,

$$\mathbf{W}_q(x/t; \mathbf{U}_i^0, \mathbf{U}_{i+1}^0) = \begin{cases} \mathbf{U}_i^0 & x < 0, \\ \mathbf{U}_{i+1}^0 & x > 0, \end{cases} \tag{5.39}$$

thus the state $\mathbf{U}_i^1 = \mathbf{U}_i^0$ is at equilibrium (i.e. $\mathcal{T} = \tau$) and is a piecewise constant stationary solution for system (5.16) and thus $\mathbf{u}_i^1 = \mathbf{u}_i^0$ is a discrete equilibrium for the resulting scheme, i.e. a stationary solution with $u = 0$ and $\frac{\Delta p}{\Delta x} = \varrho g + \mathcal{O}(\Delta x)\square$

The scheme is well balanced in the sense that it preserves a (non constant) equilibrium for (5.3) which is discretized in a natural way. This is equivalent to the strongly well balanced property of the scheme derived in section 4.3 (in the sense of definition 3.6).

Remark 5.3. This property is derived directly from the nature of the exact Godunov solver by noticing that (non constant) equilibria for system (5.16) with $u = 0$

thus $\psi(0) = \varphi(0) = 0$, satisfy $\partial_x \Pi - g\partial_x q = 0$; the solution at the evolution step satisfies Rankine-Hugoniot condition $[\Pi] = g[q]$ which means that the solution stays stationary, and this property remains valid after projection. □

Similarly, the scheme is naturally asymptotic preserving, at least from a heuristic point of view. Indeed, the scaling is only active in the evolution step, not in the projection (instantaneous relaxation) step, and this step uses an exact solver for a differential system which mimics the original system (Euler friction with potential); then, as we have seen in section 5.2, the asymptotic behavior of the original system (5.7) is preserved. The advection equations are invariant: $\partial_t \theta + u\partial_x \theta = 0$, where θ is any quantity advected by the flow, becomes $\partial_s \theta + v\partial_x \theta = 0$, the momentum equation becomes (at the order 0 in $\frac{1}{\beta}$) $\partial_x \Pi = (g - \varphi(v))\partial_x q$. Thus, from a formal continuity argument, if the solutions of (5.3) tend to solutions of (5.12) as $\beta \to \infty$, similarly, the solutions of (5.16) tend to those of the following system

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) & = 0 \\ \partial_x \Pi - (g - \varphi(v))\partial_x q & = 0 \\ \partial_s(\varrho\eta) + \partial_x(\varrho\eta v) & = 0 \\ \partial_s(\varrho\mathcal{T}) + \partial_x(\varrho\mathcal{T} v) & = 0 \\ \partial_s(\varrho q) + \partial_x(\varrho v q) & = 0, \end{cases} \tag{5.40}$$

or for the energy equation

$$\partial_s \varrho\tilde{\varepsilon} + \partial_x((\varrho\tilde{\varepsilon} + \Pi)v) - (gv - \psi(v))\partial_x q = 0. \tag{5.41}$$

For the relaxation process, we also scale the relaxation parameter $\mu = \beta\nu$ in (5.13) so that the relaxation term writes $\mu\varrho(\tau - \mathcal{T})$. Thus after first reconstruction by piecewise constant for Godunov'scheme and the projection-relaxation on the 'equilibrium' manifold $\mathcal{T} = \tau$ for the global solver, we get (after scaling) as $\beta \to \infty$ a scheme consistent with (2.22).

6. Numerical results

We present several numerical experiments, assuming linear friction and $a = 1$ for (2.1). We compare the asymptotic preserving scheme (AP) developed in the previous sections with two other schemes.

The first numerical scheme relies on a splitting method (noted SP) based on two steps. The first step consists in approximating the first order part of the system and the second one corresponds to the approximation of the source terms. The numerical method for the first step is the approximate Godunov scheme presented before in the homogeneous case ($\alpha = g = 0$). The source terms are solved by the implicit Euler method. Then, if $\mathbf{U}_i^{n+1/2}$ denotes the approximate solution after the first step, the approximate solution at the end of the time step, that is $\mathbf{U}_i^{n+1}$, is

given by:

$$\begin{cases} \varrho_i^{n+1} = \varrho_i^{n+1/2}, \\ (\varrho u)_i^{n+1} = \varrho_i^{n+1} \dfrac{u_i^{n+1/2} + g\Delta t}{1+\alpha\Delta t}, \\ (\varrho E)_i^{n+1} = (\varrho E)_i^{n+1/2} + \Delta t\big((\varrho u)_i^{n+1} g - \alpha \dfrac{((\varrho u)_i^{n+1})^2}{\varrho_i^{n+1}}\big). \end{cases} \tag{6.1}$$

This splitting scheme is neither well-balanced nor asymptotic preserving. If we start from an equilibrium data (2.9), it is naturally discretized by $u_i^0 = 0$ and some discretization of $\partial_x p = \varrho g$ of the form $\frac{\Delta p}{\Delta x} = \varrho g$. But the first step of the splitting scheme introduces waves, and an intermediate state with non null velocity (given by the value of $u^* = -\Delta p/2C$, following formulas (3.30) where $\alpha = g = 0$ for this first step), and the second step, which writes $u_i^{n+1} = \frac{u_i^{n+1/2}+g\Delta t}{1+\alpha\Delta t}$, does not restore exact null velocity. Then we note that this last formula is not invariant under the scaling $v = \alpha u, \Delta t = \alpha \Delta s$.

The second scheme (which we denote below by NP) is very close to the asymptotic preserving scheme (AP). Indeed, the only difference with the asymptotic preserving scheme AP is the definition of $\tilde{u}$, value of the velocity in the source term, in order to show the importance of the choice $\tilde{u} = u^*$ for the asymptotic preserving property, as enlightened in Remark 3.5. We thus set $\tilde{u} = u^* + \Delta p/(2C)$, which yields

$$u^* = \left(u_a - \frac{g\Delta m}{2C}\right)\left(1 + \frac{\alpha \Delta m}{2C}\right)^{-1} - \frac{1}{2C}\Delta p.$$

The numerical scheme based on this choice is strongly well-balanced but it is not asymptotic preserving (as can be easily shown, at least in Lagrangian coordinates (cf. Remark 3.5).

In the following, we present two numerical experiments. The first one corresponds to the ability of the scheme to converge to a stationary state with $u = 0$, in order to investigate the well-balanced property of the numerical schemes. The second test case illustrates the behavior of the numerical schemes for a solution mainly governed by the friction effects.

The length of the domain of simulation is 1 meter. The Courant number is 0.95 for all simulations. The equation of state is a classical ideal gas polytropic law $p = (\gamma - 1)\varrho\epsilon$, with $\gamma = 1.4$ and the gravity constant is set to 9.81 m/s^2 for both cases.

6.1. *Convergence in time to a stationary state with a null velocity*

In this test, the initial condition is composed by two constant states. The friction coefficient α is equal to 10^4 s^{-1}. In the left half of the domain, we set $(\varrho, u, p) = (2, 0, 10000)$ and in the right part of the domain, we set $(\varrho, u, p) = (1, 0, 5000)$. At the boundaries, we impose a wall boundary condition, using the classical mirror state technique. This test case enables us to investigate the long time behavior of the

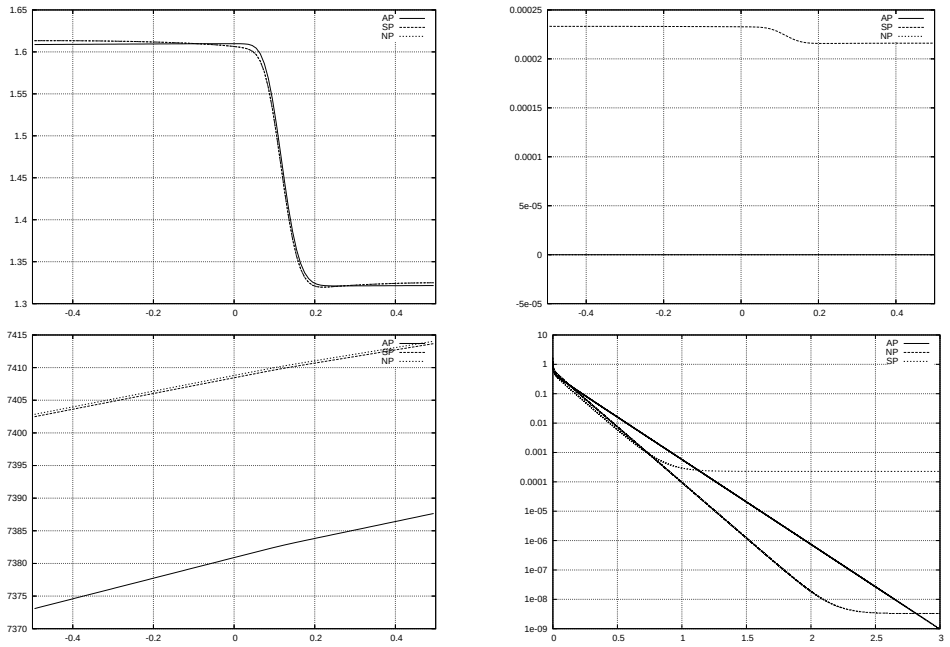

Fig. 1. Convergence to a stationary state with a null velocity: ϱ, u, p wrt. x for the converged solutions and $||u||_{\mathbb{L}^2}$ wrt. t (bottom, right)

three methods. All the figures are plotted at $t = 3$ s, which corresponds to converged (in time) solutions. The mesh is made of 100 cells. Actually, one may conclude that the three methods provide satisfactory results and we can see in Figure 1 that the results of the schemes AP and NP are very close. The main difference is noticeable when comparing with the splitting method SP for the variables u and p (note that the scaling of the plot for p has emphasized the difference). In particular, one may remark that the velocity computed by this method is small but not null. This is made obvious on the last figure were we have plotted the evolution of the (log)$\mathbb{L}^2$ norm of the velocity: for the splitting method, after some decade of decrease, the convergence history exhibits a plateau.

6.2. *Sensitivity with respect to the mesh size for large friction*

Here, we focus on the comparison of the three methods on an unsteady test case with a large friction ($\alpha = 10^6$ s^{-1}). The initial condition corresponds to a centered arch function. At the left and at the right, $(\varrho, u, p) = (1, 0, 10000)$ and in the center, $(\varrho, u, p) = (2, 0, 26390.2)$. The boundary conditions are periodic and the final time is $t = 0.01$ s, in order to obtain a Darcy-like solution. Figures 2, 3 and 4 respectively represent the results of the scheme AP, SP and NP, for several meshes (100, 1000 and 10000 cells).

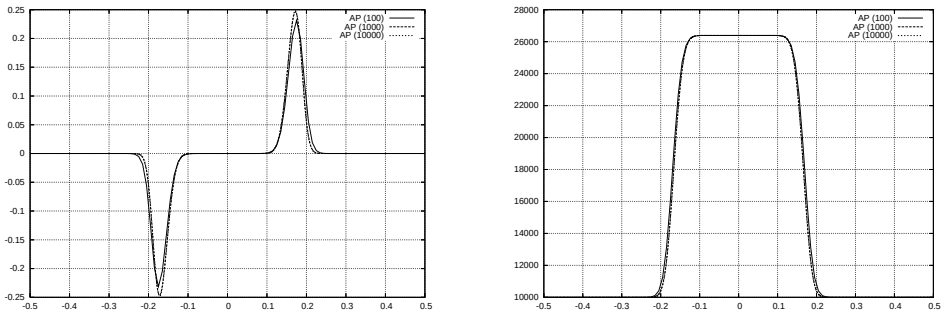

Fig. 2. Mesh convergence of the AP scheme: u and p

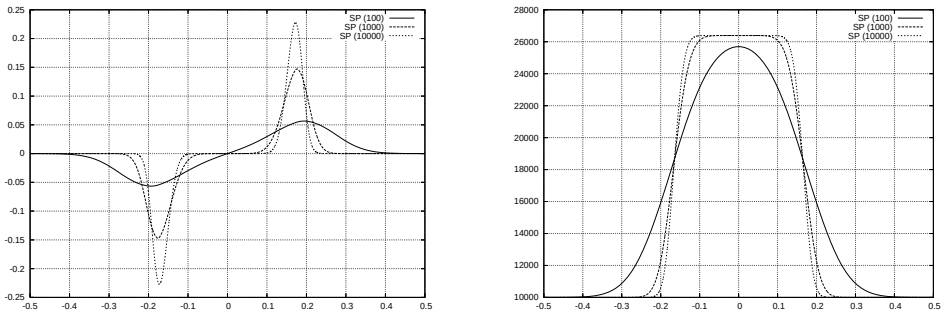

Fig. 3. Mesh convergence of the splitting method (SP): u and p

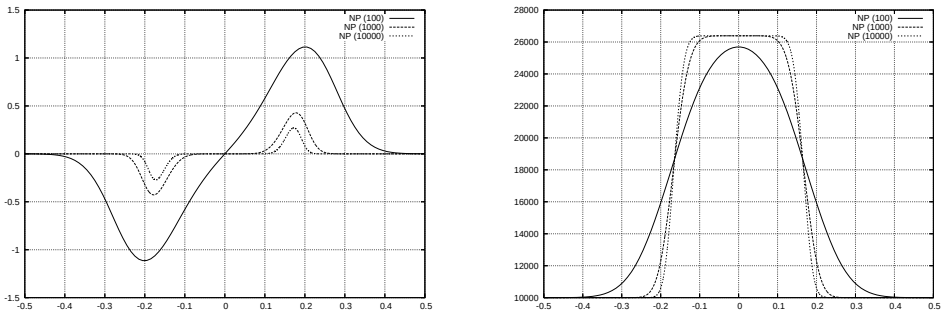

Fig. 4. Mesh convergence of the non asymptotic preserving scheme (NP): u and p

We can easily see that the results of the splitting method and of the NP scheme are very dependent on the size of the mesh (note that, for coarse meshes, the splitting method under-estimates the velocity while the NP scheme over-estimates

it). The asymptotic preserving scheme (AP) provides results in agreement with the previous analysis: its accuracy depends very little on the mesh size, which gives a hint that the numerical diffusion of the AP scheme is independent of α, contrary to the other schemes (SP and NP).

Note that, though the sign of the velocity and thus of the energy source term changes, the computations are very stable, illustrating the nice stability property of the scheme.

7. Conclusion

We have developed a methodology to derive well-balanced and asymptotic preserving schemes which is detailed on the model problem of Euler sytem with gravity and friction and can be extended to a wider class of problems. In this context, all the notions have been given a precise definition, even if the asymptotic results are not yet proved at the continuous level for the full Euler system with energy. Such properties may be crucial in specific situations, as has already been mentionned in references concerning well-balanced schemes (see for example Ref. 9). The AP property is more recently studied and has not yet received its whole interest in the applications, but this should come shortly since it may avoid, or make easy, coupling procedures for simulations of flows presenting different regimes. Moreover, the numerical illustrations above prove that the corresponding schemes behave very well on some test problems.

Apppendix: proof of Proposition 4.1

Skipping the time superscripts, the numerical scheme writes

$$\bar{\mathbf{U}}_j = \mathbf{U}_j - \frac{\Delta t}{\Delta x}(\mathbf{G}_{j+\frac{1}{2}} - \mathbf{G}_{j-\frac{1}{2}}) + \frac{\Delta t}{2}(\mathbf{S}_{j-\frac{1}{2}} + \mathbf{S}_{j+\frac{1}{2}}) \tag{7.1}$$

with

$$\mathbf{G}_{j+\frac{1}{2}} = \mathbf{G}(\mathbf{U}_j, \mathbf{U}_{j+1}), \quad \mathbf{S}_{j+\frac{1}{2}} = \widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_j, \mathbf{U}_{j+1}) \tag{7.2}$$

and

$$\begin{cases} \mathbf{G}(\mathbf{U}_L, \mathbf{U}_R) = \frac{1}{2}\big(\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R)\big) \\ -\frac{1}{2}\big\{|\lambda_1|(\mathbf{U}_L^* - \mathbf{U}_L) + |\lambda_2|(\mathbf{U}_R^* - \mathbf{U}_L^*) + |\lambda_3|(\mathbf{U}_R - \mathbf{U}_R^*)\big\}. \end{cases} \tag{7.3}$$

Recall that, as in subsection 3.3.1, we have chosen $\tilde{u} = u^*$, leading to the nonlinear scalar equation (3.39) for the actual calculation of u^*.

Under the change of variables (3.43), $\Delta t = \beta \Delta s$, we get

$$\bar{\mathbf{U}}_j = \mathbf{U}_j - \beta\frac{\Delta s}{\Delta x}(\mathbf{G}_{j+\frac{1}{2}} - \mathbf{G}_{j-\frac{1}{2}}) + \beta\frac{\Delta s}{2}(\mathbf{S}_{j-\frac{1}{2}} + \mathbf{S}_{j+\frac{1}{2}}) \tag{7.4}$$

so that the asymptotic behavior of $\beta\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$ and $\beta\widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R)$ when β (or α)$\to +\infty$ has to be determined. We first rescale the velocity, setting $u^* = \frac{1}{\beta}v^*$

and in view of (3.44), we know that

$$v^* = \varphi^{-1}(-\frac{\Delta p}{\varrho^*(\mathbf{U}_L, \mathbf{U}_R)\Delta x} + g) + \mathcal{O}(\frac{1}{\beta}) \tag{7.5}$$

while

$$\begin{cases} \lambda_1 = \dfrac{1}{\beta} v_L - C\tau_L = -\dfrac{C}{\varrho_L} + \mathcal{O}(\dfrac{1}{\beta}), \\ \lambda_2 = \dfrac{1}{\beta} v^*, \\ \lambda_3 = \dfrac{1}{\beta} v_R + C\tau_R = \dfrac{C}{\varrho_R} + \mathcal{O}(\dfrac{1}{\beta}). \end{cases} \tag{7.6}$$

Let us start with the first component of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$, that is

$$\frac{1}{2}\big\{\varrho_L u_L + \varrho_R u_R - |\lambda_1|(\varrho_L^* - \varrho_L) - |\lambda_2|(\varrho_R^* - \varrho_L^*) - |\lambda_3|(\varrho_R - \varrho_R^*)\big\}.$$

We deduce from (4.25)

$$\varrho_L^* = \varrho_L \left(1 + \frac{\varrho_L(u^* - u_L)}{C}\right)^{-1} = \varrho_L \left(1 + \frac{\varrho_L(v^* - v_L)}{\beta C}\right)^{-1} = \varrho_L + \mathcal{O}(\frac{1}{\beta})$$

and

$$\varrho_R^* = \varrho_R \left(1 - \frac{\varrho_R(u^* - u_R)}{C}\right)^{-1} = \varrho_R \left(1 - \frac{\varrho_R(v^* - v_R)}{\beta C}\right)^{-1} = \varrho_R + \mathcal{O}(\frac{1}{\beta})$$

and then

$$\varrho_L - \varrho_L^* = \varrho_L \frac{\varrho_L(v^* - v_L)}{\beta C} \left(1 + \frac{\varrho_L(v^* - v_L)}{\beta C}\right)^{-1} = \frac{\varrho_L}{\beta} \frac{\varrho_L(v^* - v_L)}{C} + \mathcal{O}(\frac{1}{\beta^2})$$

and

$$\varrho_R^* - \varrho_R = \varrho_R \frac{\varrho_R(v^* - v_R)}{\beta C} \left(1 - \frac{\varrho_R(v^* - v_R)}{\beta C}\right)^{-1} = \frac{\varrho_R}{\beta} \frac{\varrho_R(v^* - v_R)}{C} + \mathcal{O}(\frac{1}{\beta^2}).$$

The first component of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$ thus writes

$$\frac{1}{2\beta}\big\{\varrho_L v_L + \varrho_R v_R + \varrho_L(v^* - v_L) - |v^*|(\varrho_R - \varrho_L) + \varrho_R(v^* - v_R)\big\} + \mathcal{O}(\frac{1}{\beta^2})$$

$$= \frac{1}{2\beta}\big\{\varrho_L(v^* + |v^*|) + \varrho_R(v^* - |v^*|)\big\} + \mathcal{O}(\frac{1}{\beta^2})$$

$$= \frac{1}{\beta}(\varrho_L v_+^* + \varrho_R v_-^*) + \mathcal{O}(\frac{1}{\beta^2})$$

where

$$v_+^* = \max(v^*, 0) = \frac{1}{2}(v^* + |v^*|), \quad v_-^* = \min(v^*, 0) = \frac{1}{2}(v^* - |v^*|).$$

Let us carry on with the second component of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$, that is

$$\tfrac{1}{2}\big\{\varrho_L u_L^2 + p_L + \varrho_R u_R^2 + p_R - |\lambda_1|(\varrho_L^* u^* - \varrho_L u_L) - |\lambda_2|(\varrho_R^* - \varrho_L^*)u^*$$

$$-|\lambda_3|(\varrho_R u_R - \varrho_R^* u^*)\big\}.$$

We have

$$\varrho_L^* u^* - \varrho_L u_L = \frac{1}{\beta}\varrho_L \left(v^* \left(1 + \frac{\varrho_L(v^* - v_L)}{\beta C} \right)^{-1} - v_L \right)$$

$$= \frac{\varrho_L}{\beta}(v^* - v_L)\left(1 - \frac{\varrho_L v_L}{\beta C}\right)\left(1 + \frac{\varrho_L(v^* - v_L)}{\beta C}\right)^{-1}$$

$$= \frac{1}{\beta}\varrho_L(v^* - v_L) + \mathcal{O}(\frac{1}{\beta^2}),$$

then

$$(\varrho_R^* - \varrho_L^*)u^* = \frac{1}{\beta}(\varrho_R - \varrho_L)v^* + \mathcal{O}(\frac{1}{\beta^2})$$

and

$$\varrho_R u_R - \varrho_R^* u^* = \frac{1}{\beta}\varrho_R \left(v_R - v^* \left(1 - \frac{\varrho_R(v^* - v_R)}{\beta C} \right)^{-1} \right)$$

$$= \frac{\varrho_R}{\beta}(v_R - v^*)\left(1 + \frac{\varrho_R v_R}{\beta C}\right)\left(1 - \frac{\varrho_R(v^* - v_R)}{\beta C}\right)^{-1}$$

$$= \frac{1}{\beta}\varrho_R(v_R - v^*) + \mathcal{O}(\frac{1}{\beta^2}).$$

This second component of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$ thus writes

$$\frac{1}{2}\left(\frac{1}{\beta^2}\varrho_L v_L^2 + p_L + \frac{1}{\beta^2}\varrho_R v_R^2 + p_R\right)$$

$$-\frac{1}{2}\left\{\frac{C}{\beta}(v^* - v_L) + \frac{1}{\beta^2}(\varrho_R - \varrho_L)|v^*|v^* + \frac{C}{\beta}(v_R - v^*) + \mathcal{O}(\frac{1}{\beta^2})\right\} =$$

$$= p_a - \frac{C}{2\beta}\Delta v + \mathcal{O}(\frac{1}{\beta^2}).$$

At last, we focus on the third component of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$, that is

$$\tfrac{1}{2}\big\{(\varrho_L e_L + p_L)u_L + (\varrho_R e_R + p_R)u_R - |\lambda_1|(\varrho_L^* e_L^* - \varrho_L e_L) - |\lambda_2|(\varrho_R^* e_R^* - \varrho_L^* e_L^*)$$

$$-|\lambda_3|(\varrho_R e_R - \varrho_R^* e_R^*)\big\}.$$

Using (3.34) with $\Delta m = \varrho^*(\mathbf{U}_L, \mathbf{U}_R)\Delta x$ we can write

$$\varrho_L^* e_L^* - \varrho_L e_L = \varrho_L^*(e_L^* - e_L) + (\varrho_L^* - \varrho_L)e_L$$

$$= \big(\varrho_L + \mathcal{O}(\frac{1}{\beta})\big)\frac{1}{\beta C}\left(p_L v_L - p^* v^* + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big)\right)$$

$$- \left(\frac{\varrho_L}{\beta}\frac{\varrho_L(v^* - v_L)}{C} + \mathcal{O}(\frac{1}{\beta^2})\right)(\varepsilon_L + \frac{1}{2\beta^2}v_L^2)$$

$$= \frac{1}{\beta}\frac{\varrho_L}{C}\left\{p_L v_L - p^* v^* + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big) - \varrho_L(v^* - v_L)\varepsilon_L\right\} + \mathcal{O}(\frac{1}{\beta^2}),$$

then

$$\varrho_L^* e_L^* = \big(\varrho_L + \mathcal{O}(\frac{1}{\beta})\big)\left\{\varepsilon_L + \frac{1}{2\beta^2}v_L^2 + \frac{1}{\beta C}\left(p_L v_L - p^* v^* + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big)\right)\right\}$$

$$= \varrho_L\varepsilon_L + \mathcal{O}(\frac{1}{\beta}),$$

$$\varrho_R^* e_R^* = \big(\varrho_R + \mathcal{O}(\frac{1}{\beta})\big)\left\{\varepsilon_R + \frac{1}{2\beta^2}v_R^2 + \frac{1}{\beta C}\left(p^* v^* - p_R v_R + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big)\right)\right\}$$

$$= \varrho_R\varepsilon_R + \mathcal{O}(\frac{1}{\beta})$$

and finally

$$\varrho_R e_R - \varrho_R^* e_R^* = \varrho_R^*(e_R - e_R^*) + (\varrho_R - \varrho_R^*)e_R$$

$$= -\big(\varrho_R + \mathcal{O}(\frac{1}{\beta})\big)\frac{1}{\beta C}\left(p^* v^* - p_R v_R + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big)\right)$$

$$- \left(\frac{\varrho_R}{\beta}\frac{\varrho_R(v^* - v_R)}{C} + \mathcal{O}(\frac{1}{\beta^2})\right)(\varepsilon_R + \frac{1}{2\beta^2}v_R^2)$$

$$= \frac{1}{\beta}\frac{\varrho_R}{C}\left\{p_R v_R - p^* v^* - \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big) - \varrho_R(v^* - v_R)\varepsilon_R\right\} + \mathcal{O}(\frac{1}{\beta^2}).$$

The third component of $\mathbf{G}(\mathbf{U}_L,\mathbf{U}_R)$ thus equals

$$\frac{1}{2\beta}\left\{(\varrho_L(\varepsilon_L + \frac{1}{2\beta^2}v_L^2) + p_L)v_L + (\varrho_R(\varepsilon_R + \frac{1}{2\beta^2}v_R^2) + p_R)v_R\right.$$

$$- \left(p_L v_L - p^* v^* + \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big) - \varrho_L(v^* - v_L)\varepsilon_L + |v^*|(\varrho_R\varepsilon_R - \varrho_L\varepsilon_L)\right.$$

$$\left.\left. + p_R v_R - p^* v^* - \frac{\Delta x}{2}\varrho^*(\mathbf{U}_L,\mathbf{U}_R)\big((gv^* - \psi(v^*)\big) - \varrho_R(v^* - v_R)\varepsilon_R\right)\right\} + \mathcal{O}(\frac{1}{\beta^2})$$

$$= \frac{1}{2\beta}\big\{2p^* v^* + \varrho_L\varepsilon_L(v^* + |v^*|) + \varrho_R\varepsilon_R(v^* - |v^*|)\big\} + \mathcal{O}(\frac{1}{\beta^2})$$

$$= \frac{1}{\beta}(p^* v^* + \varrho_L \varepsilon_L v_+^* + \varrho_R \varepsilon_R v_-^*) + \mathcal{O}(\frac{1}{\beta^2}).$$

Gathering together the studies of the three components of $\mathbf{G}(\mathbf{U}_L, \mathbf{U}_R)$ allows to write

$$\beta \mathbf{G}(\mathbf{U}_L, \mathbf{U}_R) = \begin{pmatrix} \varrho_L v_+^* + \varrho_R v_-^* \\ \\ \beta p_a - \dfrac{C}{2}(v_R - v_L) \\ \\ p^* v^* + \varrho_L \varepsilon_L v_+^* + \varrho_R \varepsilon_R v_-^* \end{pmatrix} + \mathcal{O}(\frac{1}{\beta}). \tag{7.7}$$

On the other hand we have

$$\beta \widetilde{\mathbf{S}}(\Delta x, \Delta t; \mathbf{U}_L, \mathbf{U}_R) = \varrho^*(\mathbf{U}_L, \mathbf{U}_R) \begin{pmatrix} 0 \\ \\ \beta\big(g - \varphi(v^*)\big) \\ \\ g v^* - \psi(v^*) \end{pmatrix}. \tag{7.8}$$

We thus deduce that in the limit $\alpha \to \infty$, the numerical scheme (7.4) goes to

$$\begin{cases} \bar{\varrho}_j = \varrho_j - \dfrac{\Delta s}{\Delta x}\Big\{(\varrho_j v_{j+\frac{1}{2}}^+ + \varrho_{j+1} v_{j+\frac{1}{2}}^-) - (\varrho_{j-1} v_{j-\frac{1}{2}}^+ + \varrho_j v_{j-\frac{1}{2}}^-)\Big\}, \\ \\ \dfrac{1}{\Delta x}(p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}) = \dfrac{1}{2}\Big\{\varrho_{j-\frac{1}{2}}(g - \varphi(v_{j-\frac{1}{2}})) + \varrho_{j+\frac{1}{2}}(g - \varphi(v_{j+\frac{1}{2}}))\Big\}, \\ \\ \bar{\varrho}_j \bar{\varepsilon}_j = \varrho_j \varepsilon_j - \dfrac{\Delta s}{\Delta x}\Big\{p_{j+\frac{1}{2}} v_{j+\frac{1}{2}} - p_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \\ \\ +(\varrho_j \varepsilon_j v_{j+\frac{1}{2}}^+ + \varrho_{j+1}\varepsilon_{j+1} v_{j+\frac{1}{2}}^-) - (\varrho_{j-1}\varepsilon_{j-1} v_{j-\frac{1}{2}}^+ + \varrho_j \varepsilon_j v_{j-\frac{1}{2}}^-)\Big\} \\ \\ +\dfrac{\Delta s}{2}\varrho_{j+\frac{1}{2}}\Big(g(v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}) - \big(\psi(v_{j-\frac{1}{2}}) + \psi(v_{j+\frac{1}{2}})\big)\Big) \end{cases} \tag{7.9}$$

where

$$\begin{cases} \varrho_{j+\frac{1}{2}} = \varrho^*(\mathbf{U}_j, \mathbf{U}_{j+1}), \\ \\ v_{j+\frac{1}{2}} = \varphi^{-1}\left(g - \dfrac{1}{\varrho_{j+\frac{1}{2}}}\dfrac{p_{j+1} - p_j}{\Delta x}\right), \\ \\ p_{j+\frac{1}{2}} = \dfrac{1}{2}(p_j + p_{j+1}). \end{cases} \tag{7.10}$$

The pair $(\bar{\varrho}_j, \bar{\varepsilon}_j)$ is then updated by

$$\begin{cases} \bar{\varrho}_j = \varrho_j - \frac{\Delta s}{\Delta x}\left\{(\varrho_j v^+_{j+\frac{1}{2}} + \varrho_{j+1} v^-_{j+\frac{1}{2}}) - (\varrho_{j-1} v^+_{j-\frac{1}{2}} + \varrho_j v^-_{j-\frac{1}{2}})\right\}, \\ \bar{\varrho}_j\bar{\varepsilon}_j = \varrho_j\varepsilon_j - \frac{\Delta s}{\Delta x}\left\{(\varrho_j\varepsilon_j v^+_{j+\frac{1}{2}} + \varrho_{j+1}\varepsilon_{j+1} v^-_{j+\frac{1}{2}}) - (\varrho_{j-1}\varepsilon_{j-1} v^+_{j-\frac{1}{2}} + \varrho_j\varepsilon_j v^-_{j-\frac{1}{2}})\right. \\ \qquad + p_{j+\frac{1}{2}} v_{j+\frac{1}{2}} - p_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \\ \qquad \left. + \frac{\Delta s}{2}\varrho_{j+\frac{1}{2}}\Big(g(v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}) - \big(\psi(v_{j-\frac{1}{2}}) + \psi(v_{j+\frac{1}{2}})\big)\Big)\right. \end{cases} \tag{7.11}$$

with $\varrho_{j+\frac{1}{2}}, v_{j+\frac{1}{2}}$ and $p_{j+\frac{1}{2}}$ given by (7.10). Conversely, (7.10)-(7.11) imply (7.9).

Formulas (7.11) actually define a consistent explicit numerical scheme of the asymptotic system

$$\begin{cases} \partial_s\varrho + \partial_x\varrho v = 0, \\ \partial_s\varrho\varepsilon + \partial_x(\varrho\varepsilon + p)v = \varrho\big(gv - \psi(v)\big) \end{cases} \tag{7.12}$$

with

$$\begin{cases} v = \varphi^{-1}(g - \dfrac{1}{\varrho}\partial_x p), \\ p = p(\varrho, \varepsilon), \end{cases} \tag{7.13}$$

which gives the desired result.

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