NON-MONOTONIC TRAVELING WAVES IN VAN DER WAALS FLUIDS

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Abstract. We investigate the existence and properties of traveling wave solutions for the hyperbolic-elliptic system of conservation laws describing the dynamics of van der Waals fluids. The model is based on a constitutive equation of state containing two inflection points and incorporates nonlinear viscosity and capillarity terms. A global description of the traveling wave solutions is provided. We distinguish between classical and nonclassical trajectories and, for the latter, the existence and properties of kinetic functions is investigated. An earlier work in this direction (cf. N. Bedjaoui and P.G. LeFloch, Diffusive-dispersive traveling waves and kinetic relations II. A hyperbolic-elliptic model of phase transitions dynamics, Proc. Royal Soc. Edinburgh 132A (2002), 545-565.) was restricted to dealing with one inflection point only. Specifically, given any right-hand state and any shock speed (within some admissible range), we prove the existence of a nonclassical traveling wave for a *sequence of parameter values* representing the ratio of viscosity and capillarity. Our analysis exhibits a surprising lack of monotonicity of traveling waves. The behavior of these nonclassical trajectories is also investigated numerically.

1. Introduction

In this paper, we are interested in the effect of viscosity and capillarity in compressive fluids governed by the following two conservation laws:

$$\begin{aligned} &\partial_t \tau - \partial_x u = 0, \\ &\partial_t u + \partial_x p(\tau) = \alpha\, \partial_x(\beta(\tau)\, |\partial_x \tau|^q\, \partial_x u) - \partial_{xxx}\tau. \end{aligned} \tag{1.1}$$

Here, u and τ represent the velocity and the specific volume of the fluid, respectively, while β is a smooth, positive function and α is a non-negative parameter representing the strength of the viscosity, and q is a non-negative exponent. The pressure law $p = p(\tau)$ is a positive function defined for all $\tau \in (0, +\infty)$ and of the following van der Waals type:

$$\lim_{\tau\to 0} p(\tau) = +\infty, \qquad \lim_{\tau\to+\infty} p(\tau) = 0, \tag{1.2}$$

and there exist $0 < a < c$ such that

$$\begin{aligned} &p''(\tau) > 0, \quad \tau \in (0, a) \cup (c, +\infty), \\ &p''(\tau) < 0, \quad \tau \in (a, c), \\ &p'(a) > 0. \end{aligned} \tag{1.3}$$

The left-hand side of (1.1) forms a first-order system of partial differential equations, which is of elliptic type when τ belongs in the interval (d, e) characterized by the conditions $0 < d < a < e < c$ and $p'(d) = p'(e) = 0$. It is of hyperbolic type when $\tau \in (0, d) \cup (e, +\infty)$ and admits the two (distinct, real) wave speeds $\pm\sqrt{-p'(\tau)}$.

The selection of physically meaningful shock waves of the first-order system is determined by the traveling waves associated with the augmented system (1.1) with incorporates viscosity and capillarity effects, i.e. solutions depending only on the variable $y := x - \lambda t$ for some speed

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λ and connecting two constant states (τ_-, u_-) and (τ_+, u_+) at infinity. The present work is a continuation of the series of papers by Bedjaoui and LeFloch [3]–[7]. The novelty here lies in the behavior of the pressure function which admits two inflection points. We are going to establish the existence of classical and nonclassical traveling waves for the model (1.1)–(1.3), and to investigate their properties. For background on nonclassical shock waves for systems of conservation laws we refer to the monograph LeFloch [20].

The first mathematical works on van der Waals fluids go back to Slemrod [25, 26], who investigated self-similar approximations of the Riemann problem. The concept of a kinetic relation associated with (undercompressive) nonclassical shocks and phase boundaries was introduced by Abeyaratne and Knowles [1, 2], Truskinovsky [27, 28], and LeFloch [19, 20, 21]. Traveling waves for diffusive-dispersive, scalar equations were studied by Bona and Schonbek [10], Jacobs, McKinney, and M. Shearer [18]. Nonclassical shocks were also found in thin film models; cf. Bertozzi and Shearer [9] and Otto and Westdickenberg [23]. Nonlinear hyperbolic systems were considered by Hayes and LeFloch [15, 16, 17]. See also the extensive works by Benzoni, Colombo, Corli, Fan cited for instance in [8, 11, 12, 13, 14]. See also [22, 24] and the references therein.

A traveling wave solution $y \mapsto (\tau(y), u(y))$ should satisfy

$$
\begin{aligned}
&\lambda\, \tau_y + u_y = 0,\\
&-\lambda\, u_y + p(\tau)_y = \alpha\, (\beta(\tau)\, |\tau_y|^q\, u_y)_y - \tau_{yyy},
\end{aligned}
\tag{1.4}
$$

together with the boundary conditions

$$
\begin{aligned}
&u_y(y),\ \tau_y(y),\ u_{yy}(y),\ \tau_{yy}(y) \to 0 \qquad \text{when } y \to \pm\infty,\\
&u(y) \to u_-, \quad \tau(y) \to \tau_- \qquad \text{when } y \to -\infty,\\
&u(y) \to u_+, \quad \tau(y) \to \tau_+ \qquad \text{when } y \to +\infty,
\end{aligned}
\tag{1.5}
$$

where u_-, τ_-, u_+, τ_+ are given constants. To describe the traveling wave solutions, it is convenient to set

$$
u_0 := u_-, \qquad \tau_0 := \tau_-
$$

and to search for all possible right-hand states (u_+, τ_+) that can be attained through a traveling wave.

By integration of (1.4) over some interval $(-\infty, y]$ and by using (1.5) we obtain

$$
\begin{aligned}
&\lambda\, (\tau - \tau_0) + u - u_0 = 0,\\
&\lambda\, (u - u_0) - p(\tau) + p(\tau_0) = -\alpha\, \beta(\tau) |\tau_y|^q u_y + \tau_{yy},
\end{aligned}
\tag{1.6}
$$

which consists in a second-order, ordinary differential equation and an algebraic equation. By letting $y \to \infty$ in (1.6) and using (1.5) it follows that the shock speed λ is determined by the (Rankine-Hugoniot) relation

$$
\lambda\, (\tau_+ - \tau_0) + u_+ - u_0 = -\lambda\, (u_+ - u_0) + p(\tau_+) - p(\tau_0) = 0,
$$

hence

$$
\lambda^2 = -\frac{p(\tau_+) - p(\tau_0)}{\tau_+ - \tau_0} \quad \text{if} \quad \tau_+ \neq \tau_0.
\tag{1.7}
$$

Obviously, λ must be real, which implies that $\big(p(\tau_+) - p(\tau_0)\big)\, (\tau_+ - \tau_0) \leq 0$.

Observe that we can eliminate the variable u in (1.6) and obtain an equation only in τ:

$$
-\lambda^2\, (\tau - \tau_0) - p(\tau) + p(\tau_0) = \alpha\, \lambda\, \beta(\tau) |\tau_y|^q \tau_y + \tau_{yy}.
\tag{1.8}
$$

In turn, setting $v = \tau_y$, we can reformulate (1.8) in the form of a first-order system in the two variables τ, v:

$$
\begin{aligned}
&\tau_y = v,\\
&v_y = -\lambda\, \alpha\, \beta(\tau) |v|^q v + \frac{\partial G}{\partial \tau}(\tau_0, \lambda, \tau),
\end{aligned}
\tag{1.9}
$$

where the function G is defined by

$$
G(\tau_0, \lambda, \tau) := \int_{\tau_0}^{\tau} \big(p(\tau_0) - p(s) - \lambda^2\, (s - \tau_0)\big)\, ds.
\tag{1.10}
$$

The boundary conditions (1.5) now read

$$\begin{aligned} &\tau_y(y),\, v_y(y) \to 0 \qquad \text{when } y \to \pm\infty,\\ &\tau(y) \to \tau_- = \tau_0, \quad v(y) \to 0 \qquad \text{when } y \to -\infty,\\ &\tau(y) \to \tau_+, \quad v(y) \to 0 \qquad \text{when } y \to +\infty. \end{aligned} \tag{1.11}$$

Note that the sign of the viscosity terms in (1.8) depends on the speed λ. Applying the transformation $y \mapsto -y$ if necessary, without loss of generality we can suppose that $\lambda > 0$. In other words, we focus attention on the 2-shock waves. The case $\lambda < 0$ of 1-shock wave follows immediately by exchanging the role of (u_-, τ_-) and (u_+, τ_+).

Solutions of (1.1) satisfy the following entropy inequality

$$\begin{aligned} &\partial_t\Big(U(\tau,u) + \frac{(\partial_x\tau)^2}{2}\Big) + \partial_x\Big(F(\tau,u) - \partial_t\tau\,\partial_x\tau + u\,\partial_{xx}\tau - u\,\alpha\,\beta(\tau)\,|\partial_x\tau|^q\,\partial_x u\Big)\\ &= -\alpha\,\beta(\tau)|\partial_x\tau|^q(\partial_x u)^2, \end{aligned} \tag{1.12}$$

where

$$U(\tau,u) := -\int^\tau p(s)\,ds + \frac{u^2}{2}, \qquad F(\tau,u) := u\,p(\tau).$$

In turn, traveling wave solutions satisfy the algebraic inequality

$$-\lambda\,\big(U(\tau_+,u_+) - U(\tau_-,u_-)\big) + F(\tau_+,u_+) - F(\tau_-,u_-) \le 0. \tag{1.13}$$

Throughout this paper we will not distinguish between the traveling wave (determined up a translation in the x-variable) and its orbit in the phase plane.

2. Monotonicity of the semi-orbits and dissipation-free trajectories

We fix a left-hand state $\tau_- = \tau_0 > 0$ and we consider trajectories propagating with a positive speed λ that corresponds to 2-shock waves. Henceforth, to exhibit nonclassical trajectories which is the most interesting case, we consider the geometric situation where there are exactly four intersection points $(\tau_i, p(\tau_i))$, $i = 0, .., 4$, with

$$\tau_0 < \tau_1 < \tau_2 < \tau_3,$$

of the line $\tau \mapsto p(\tau_0) - \lambda^2\,(\tau - \tau_0)$ and the curve $\tau \mapsto p(\tau)$. It is immediate to check that $(\tau_i, 0)$, $i = 0, 1, 2, 3$, are precisely the equilibrium points of the ODE system (1.9), i.e. satisfy

$$\frac{\partial G}{\partial \tau}(\tau_0, \lambda, \tau_i) = 0, \quad i = 0, 1, 2, 3. \tag{2.1}$$

For τ_0 and λ fixed, the function $G_\tau(\tau) = \frac{\partial G}{\partial \tau}(\tau_0, \lambda, \tau)$ satisfies the sign property

$$\begin{aligned} &G_\tau(\tau) > 0, \quad \tau \in (\tau_0, \tau_1) \cup (\tau_2, \tau_3),\\ &G_\tau(\tau) < 0, \quad \tau \in (0, \tau_0) \cup (\tau_1, \tau_2) \cup (\tau_3, +\infty). \end{aligned} \tag{2.2}$$

The eigenvalues of the linearized system corresponding to (1.9) at an equilibrium point are given by

$$\mu_\pm(\tau, \lambda, \alpha) = \begin{cases} \frac{1}{2}(-\alpha\lambda\beta(\tau) \pm \sqrt{(\alpha\lambda\beta(\tau))^2 - 4(p'(\tau) + \lambda^2)}), & \text{if} \quad q = 0,\\ \pm\sqrt{-p'(\tau) - \lambda^2}, & \text{if} \quad q > 0, \end{cases} \tag{2.3}$$

and the corresponding eigenvectors are simply ${}^t(1, \mu_\pm)$. The four equilibrium points satisfy

$$p'(\tau_i) + \lambda^2 < 0 \quad \text{for} \quad i = 0, 2 \qquad p'(\tau_i) + \lambda^2 > 0 \quad \text{for} \quad i = 1, 3,$$

and we see easily that:

Lemma 2.1. *For all $q \ge 0$, the equilibria $(\tau_0, 0)$ and $(\tau_2, 0)$ are saddle points (two real eigenvalues with opposite signs). On the other hand, the equilibria $(\tau_1, 0)$ and $(\tau_3, 0)$ are centers (two purely imaginary eigenvalues) if $q > 0$. For $q = 0$ and $i = 1, 3$, the point $(\tau_i, 0)$ is a stable node (two negative eigenvalues) if $p'(\tau_i) + \lambda^2 \le \frac{(\alpha\lambda\beta(\tau_i))^2}{4}$, but is a stable spiral (two eigenvalues with the same negative real part but with opposite non-zero imaginary parts) if $p'(\tau_i) + \lambda^2 > \frac{(\alpha\lambda\beta(\tau_i))^2}{4}$.*

We are looking for trajectories connecting $(\tau_0, 0)$ to one of the other equilibrium points. In particular, we are interested in nonclassical trajectories, i.e. those connecting $(\tau_0, 0)$ to $(\tau_2, 0)$.

By multiplying (1.8) by τ_y and integrating over $(-\infty, +\infty)$, we obtain that an eventual traveling wave connecting τ_0 to τ_2 necessarily satisfies

$$G(\tau_0, \lambda, \tau_2) = \int_{\tau_0}^{\tau_2} (p(\tau_0) - p(s) - \lambda^2(s - \tau_0))\, ds \geq 0. \tag{2.4}$$

In the special case $G(\tau_0, \lambda, \tau_2) = 0$, there is a connection from τ_0 to τ_2 iff $\alpha = 0$ (see [3], [4] and [7]). If $\alpha > 0$, for an eventual such connection, the inequality (2.4) must be strict, which will always be assumed from now on. Note that (2.4) is nothing but the entropy inequality already derived in (1.13).

To any of the two equilibria $(\tau_k, 0)$, $k = 0, 2$ let us associate the four quadrants

$$Q^1_{\tau_k} = \{(\tau, v)/\tau \geq \tau_k, v \geq 0\}, \quad \text{and} \quad Q^3_{\tau_k} = \{(\tau, v)/\tau \leq \tau_k, v \leq 0\}$$

in which the trajectories issuing from $(\tau_k, 0)$ lie for large and negative values of y, and

$$Q^2_{\tau_k} = \{(\tau, v)/\tau \leq \tau_k, v \geq 0\}, \quad \text{and} \quad Q^4_{\tau_k} = \{(\tau, v)/\tau \geq \tau_k, v \leq 0\}$$

in which the trajectories arriving at $(\tau_k, 0)$ lie for large and positive values of y.

Clearly, as long the derivative τ_y keeps a constant sign we can introduce the change of variable $y \to \tau$, and express the equations (1.9) directly in the phase plane :

$$v(\tau)\, \frac{dv}{d\tau}(\tau) + \alpha\, \lambda\, \beta(\tau) |v(\tau)|^q v(\tau) = G_\tau(\tau_0, \lambda, \tau). \tag{2.5}$$

For $k = 0, 2$ we denote by $\mathcal{C}^1_{\tau_k,\alpha}$ and $\mathcal{C}^3_{\tau_k,\alpha}$ the trajectories issuing from $(\tau_k, 0)$ in $Q^1_{\tau_k}$ and in $Q^3_{\tau_k}$, respectively. Similarly, we denote by $\mathcal{C}^2_{\tau_k,\alpha}$ and $\mathcal{C}^4_{\tau_k,\alpha}$ the trajectories reaching the point $(\tau_k, 0)$ and arriving in $Q^2_{\tau_k}$ and in $Q^4_{\tau_k}$, respectively.

Proposition 2.2. *Fix two viscosity coefficients $0 \leq \alpha_1 < \alpha_2$, then the trajectories arriving at and issuing from the saddle points $(\tau_k, 0)$, $k = 0, 2$ satisfy the following ordering properties :*

- *For $l = 1, 3$ and while the curves don't meet at the $\tau - axis$, the curve $\mathcal{C}^l_{\tau_k,\alpha_1}$ remains above (respectively, below) the curve $\mathcal{C}^l_{\tau_k,\alpha_2}$ in the half phase plane $\{v > 0\}$ (respectively in the half phase plane $\{v < 0\}$).*
- *For $l = 2, 4$ and while the curves don't meet at the $\tau - axis$, the curve $\mathcal{C}^l_{\tau_k,\alpha_1}$ remains below (respectively, above) the curve $\mathcal{C}^l_{\tau_k,\alpha_2}$ in the half phase plane $\{v > 0\}$ (respectively in the half phase plane $\{v < 0\}$).*

Proof. Let us for instance check the claim in the half plane $\{v > 0\}$, corresponding to $l = 1$, $k = 0$. The proof of the other cases is completely similar.

Given $\alpha_2 > \alpha_1 \geq 0$ we will first check that $\mathcal{C}^1_{\tau_0,\alpha_1}$ remains above $\mathcal{C}^1_{\tau_0,\alpha_2}$ in a neighborhood of $(\tau, v) = (\tau_0, 0)$. This property is obvious when $q = 0$ since

$$\frac{\partial \mu_+}{\partial \alpha}(\tau_k, \lambda, \alpha) < 0.$$

When $q > 0$, the proof is not immediate since the eigenvalue μ_+ (which determines the tangents to the semi orbits at an equilibrium point) satisfy $\mu_+(\tau_0, \lambda, \alpha) = \mu_+(\tau_0, \lambda, 0)$. i.e. is independent of α.

We proceed as follows. For $j = 1, 2$, we denote by $\tau \mapsto V_j(\tau)$ the function corresponding to $\mathcal{C}^1_{\tau_0,\alpha_j}$ in the half phase plane $\{v \geq 0\}$. Such function is well defined in an interval of the form $[\tau_0, \tau^j]$ where τ^j is the first value such that $V_j(\tau^j) = 0$. Then we can write

$$V_j(\tau)\, \frac{dV_j}{d\tau}(\tau) + \alpha_j\, \lambda\, \beta(\tau) |V_j(\tau)|^q V_j(\tau) = G_\tau(\tau_0, \lambda, \tau), \qquad j = 1, 2, \tag{2.6}$$

and therefore

$$\frac{1}{2}\, \frac{d}{d\tau}(V_1^2 - V_2^2) = \lambda\, \beta(\tau)(\alpha_2 V_2^{q+1} - \alpha_1 V_1^{q+1}). \tag{2.7}$$

From the asymptotic expansion

$$V_j(\tau) \sim (\tau - \tau_0)\, \mu_+(\tau_0, \lambda), \qquad \tau \to \tau_0,$$

it follows that, in a small neighborhood of $(\tau_0, 0)$,

$$\frac{1}{2}\frac{d}{d\tau}(V_1^2 - V_2^2) \sim \lambda\,\beta(\tau_0)(\alpha_2 - \alpha_1)\mu_+(\tau_0, \lambda)^{q+1}\,(\tau - \tau_0)^{q+1} > 0.$$

This shows the desired monotonicity property in a neighborhood of $(\tau_0, 0)$.

Assume now that there is a (first) value $\xi > \tau_0$ such that $V_1(\xi) = V_2(\xi) > 0$. Then, clearly $\frac{dV_1}{d\tau}(\xi) \le \frac{dV_2}{d\tau}(\xi)$. But this leads to a contradiction since by (2.6)

$$0 < (\alpha_2 - \alpha_1)\,\lambda\,\beta(\xi)\,V_1(\xi)^{q+1} = V_1(\xi)\Big(\frac{dV_1}{d\tau}(\xi) - \frac{dV_2}{d\tau}(\xi)\Big) \le 0.$$

□

For completeness we state here without proof a result for the dissipation-free case. The proof is omited.

Proposition 2.3. (Trajectories for $\alpha = 0$.)

• $\mathcal{C}^1_{\tau_0,0} = \mathcal{C}^4_{\tau_0,0}$. *Moreover, let* $\tau_{max} > \tau_3$ *be characterized by the equality area*

$$G(\tau_0, \lambda, \tau_{max}) = \int_{\tau_0}^{\tau_{max}} (p(\tau_0) - p(\tau) - \lambda^2(\tau - \tau_0))\,d\tau = 0. \tag{2.8}$$

Then in the variable $\tau = \tau(y)$ *the trajectory is monotone increasing in some interval* $(-\infty, y_{max})$ *with* $\tau(y_{max}) = \tau_{max}$ *and* $\tau_y(y_{max}) = 0$, *and monotone decreasing on* $(y_{max}, +\infty)$.

• $\mathcal{C}^1_{\tau_2,0} = \mathcal{C}^4_{\tau_2,0}$ *(respectively,* $\mathcal{C}^3_{\tau_2,0} = \mathcal{C}^2_{\tau_2,0}$*). Moreover, let us consider* $\overline{\tau} > \tau_3$ *(respectively* $\underline{\tau} < \tau_1$*) characterized by the two conditions (equality areas)*

$$\begin{aligned} G(\tau_2, \lambda, \overline{\tau}) &= \int_{\tau_2}^{\overline{\tau}} (p(\tau_0) - p(\tau) - \lambda^2(\tau - \tau_0))\,d\tau = 0, \\ G(\tau_2, \lambda, \underline{\tau}) &= \int_{\tau_2}^{\underline{\tau}} (p(\tau_0) - p(\tau) - \lambda^2(\tau - \tau_0))\,d\tau = 0. \end{aligned} \tag{2.9}$$

Then, the corresponding trajectory in the (y, τ) *plane is monotone increasing (respectively decreasing) in some interval* $(-\infty, \overline{y})$ *(respectively* $(-\infty, \underline{y})$*) where* $\tau(\overline{y}) = \overline{\tau}$ *(respectively,* $\tau(\underline{y}) = \underline{\tau}$*) and* $\tau_y(\overline{y}) = 0$ *(respectively* $\tau_y(\underline{y}) = 0$*), and monotone decreasing (respectively increasing) on* $(\overline{y}, +\infty)$ *(respectively* $(\underline{y}, +\infty)$*).*

Note that in the previous Proposition, since equation (1.8) is autonomous, we can take $\underline{y} = \overline{y} = y_{max} = 0$. Then the three trajectories cited in Proposition 2.3 become odd functions in the (y, τ) plane.

The existence of the values $\underline{\tau}$, $\overline{\tau}$ and τ_{max} is easily checked geometrically by straightforward monotonicity arguments on the graph of the pressure. In view of the condition (2.4) we have

$$\tau_0 < \underline{\tau} < \tau_1 < \tau_2 < \tau_3 < \overline{\tau} < \tau_{max} \tag{2.10}$$

3. Traveling waves with at most one oscillation

Our objective is to determine, for each parameter value $\alpha > 0$, all possible connections from τ_0 to one of the other three equilibria. Note first that the curve $\mathcal{C}^3_{\tau_0,\alpha}$ cannot connect to any equilibrium point. For otherwise, there would exist a point y_0 such that

$$\tau(y_0) < \tau_0, \qquad \tau_y(y_0) = 0, \qquad \tau_{yy}(y_0) \ge 0.$$

However, by (1.8) this immediately would lead to a contradiction since the left-hand side of (1.8) is negative while, by (2.2), the right-hand side is non-negative. Therefore, the only “interesting” trajectory is $\mathcal{C}^1_{\tau_0,\alpha}$. Let us recall here that, in Proposition 2.3 we have established that, for the extreme case $\alpha = 0$,

$$C^1_{\tau_0,0} = C^4_{\tau_0,0}.$$

Considering now the values $\alpha > 0$. Then, thanks to the specific form of the graph of the pressure function p we obtain:

Proposition 3.1. *For all* $\alpha > 0$ *the trajectory* $\mathcal{C}^1_{\tau_0,\alpha}$ *connects the point* $(\tau_0, 0)$ *to one of the equilibrium points* $(\tau_i, 0)$, $i = 1, 2, 3$.

Proof. Multiplying (1.8) by τ_y and integrating the resulting identity over an interval $(-\infty, y)$ we get

$$\frac{1}{2}\tau_y^2(y) + \alpha\,\lambda \int_{-\infty}^{y} \beta(\tau(s))|\tau_y(s)|^{q+2}\,ds = G(\tau_0, \lambda, \tau(y)). \tag{3.1}$$

But, from (2.2) and (2.8), $G(\tau_0, \lambda, \tau) \leq 0$ for $\tau \geq \tau_{max}$. This implies that $\tau \leq \tau_{max}$. Also, we have necessarily $\tau \geq \tau_0$. Finally τ is bounded and then by (3.1) τ_y is bounded too. We deduce that such trajectory necessarily converges to one of the equilibrium points $(\tau_k, 0)$, $k = 1, 2, 3$. $(\tau_0, 0)$ is excluded by (3.1) since $\alpha > 0$ and $G(\tau_0, \lambda, \tau_0) = 0$. □

Note that in the case of a single inflection point, the previous result is false for small values of α for which there is no connection from τ_0 to any equilibria (see [7], [3]and [4]).

We will now investigate the nature of the connections by varying the parameter α in the whole interval $(0, +\infty)$. At this juncture it is convenient to introduce the following definition:

Definition 3.2. *A traveling wave solution $\tau = \tau(y)$ of (1.8) connecting τ_0 to τ_2 is said to have n* **oscillations** *if there exists exactly n points $y_1 < \ldots < y_n$ such that the function τ is alternatively strictly monotone increasing and strictly monotone decreasing in the intervals delimited by these points.*

For instance, by definition a monotone traveling wave has 0 oscillation. We begin with the existence of traveling waves for sufficiently large diffusion.

Theorem 3.3. • Monotone trajectories. *Given τ_0 and $\lambda > 0$, there exists a unique value $\alpha = \overline{\alpha}_0(\tau_0, \lambda) > 0$ for which there is a unique* monotone *traveling wave connecting τ_0 to τ_2.*

• *For larger values of the diffusion parameter, that is for $\alpha > \overline{\alpha}_0(\tau_0, \lambda)$ there exists a connection from τ_0 to τ_1.*

We omit the proof of this result, since it follows exactly as in [3] for $q = 0$ and in [7] for $q > 0$. Those papers restricted attention to flux functions (here the pressure) with a single inflection point which is not the case here (see also [4] and [5]). But, for such trajectories, τ lies in the interval (τ_0, τ_2), and all of the arguments developed in [3, 7] do not take account of the behavior of the flux function in the region $\tau > \tau_2$; therefore, the results therein remain valid for our situation.

Note that the proof of the first item was based on monotony and continuity arguments of the semi orbits with respect to α, for the uniqueness of $\overline{\alpha}_0$, while the existence of such value is essentially due to condition (2.4). The proof of the second item is due to monotony arguments.

Now, we are interested in the case $0 < \alpha < \overline{\alpha}_0$. For such value α, we denote by $(\hat{\tau}_n(\tau_0, \alpha), 0)$, the n^{th} eventual intersection point of $\mathcal{C}^1_{\tau_0,\alpha}$ and the τ-axis. Concerning the trajectories arriving at $(\tau_2, 0)$ we denote by $\check{\tau}_+(\tau_2, \alpha)$ (respectively $\check{\tau}_-(\tau_2, \alpha)$), the last intersection point of $\mathcal{C}^4_{\tau_2,\alpha}$, (respectively $\mathcal{C}^2_{\tau_2,\alpha}$) and the τ-axis. We define also $\hat{\tau}_+(\tau_2, \alpha)$ (respectively $\hat{\tau}_-(\tau_2, \alpha)$), the first intersection point of $\mathcal{C}^1_{\tau_2,\alpha}$, (respectively $\mathcal{C}^3_{\tau_2,\alpha}$) and the τ-axis. Then we obtain:

Theorem 3.4. • 1-oscillation trajectories. *There exists a unique value $\alpha = \overline{\alpha}_1(\tau_0, \lambda)$, $0 < \overline{\alpha}_1(\tau_0, \lambda) < \overline{\alpha}_0(\tau_0, \lambda)$, for which there is a unique traveling wave with a single oscillation, connecting τ_0 to τ_2.*

• *For $\alpha \in (\overline{\alpha}_1, \overline{\alpha}_0)$ there is a connection between τ_0 and τ_3.*

This is one of our key result in the present paper. Recall that (see ref [3]-[7]) for pressure functions with a single inflection point, all trajectories corresponding to saddle-saddle connections are monotone. We will now provide the proof of Theorem 3.5 split into several propositions and lemmas.

Proposition 3.5.

• *The function $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$ is continuous and monotone decreasing on $[0, \overline{\alpha}_0)$ satisfying*

$$\lim_{\alpha \mapsto \overline{\alpha}_0} \hat{\tau}_1(\tau_0, \alpha) = \hat{\tau}_+(\tau_2, \overline{\alpha}_0). \tag{3.2}$$

• *The function* $\alpha \mapsto \check{\tau}_+(\tau_2, \alpha)$ *is continuous and* strictly *monotone increasing in some maximal interval of the form* $[0, \alpha_{max})$ *where* $\alpha_{max} \le +\infty$ *and* $\alpha_{max} < +\infty$ *implies* $\lim_{\alpha \to \alpha_{max}} \check{\tau}_+(\tau_2, \alpha) = +\infty$.

Proof. First the monotonicity of the functions $\hat{\tau}_1(\tau_0, \alpha)$ and $\check{\tau}_+(\tau_2, \alpha)$ with respect to α is a direct consequence of Proposition 2.2. The *strict* monotonicity of $\check{\tau}_+$ is proved as follows:

Suppose that there exist $0 \le \alpha_1 < \alpha_2$ such that $\check{\tau}_+(\tau_2, \alpha_1)$ and $\check{\tau}_+(\tau_2, \alpha_2)$ exist and $\check{\tau}_+(\tau_2, \alpha_1) = \check{\tau}_+(\tau_2, \alpha_2) = \xi$. Then, by assumptions, $\tau_3 < \overline{\tau} \le \xi$. Now, setting $(\tau, V_j(\tau))$ the varying point representing $C^4_{\tau_2, \alpha_j}$ for $j = 1, 2$, then both functions satisfy (2.6). By Proposition 2.2, we have in one hand

$$V_2(\tau) < V_1(\tau) < 0, \qquad \tau \in (\tau_2, \xi). \tag{3.3}$$

On the other hand, using (2.2), the two corresponding functions in the plane (y, τ), up to some translation, satisfy (1.8) with the initial conditions:

$$\tau(0) = \xi, \quad \tau_y(0) = 0, \quad \text{and} \quad \tau_{yy}(0) = G_\tau(\tau_0, \lambda, \xi) = \kappa < 0.$$

We obtain that in the left neighborhood of $(\xi, 0)$ and for $j = 1, 2$,

$$V_j(\tau) \sim -\sqrt{2\,|\kappa|(\xi - \tau)}.$$

Then, from (2.6) we get

$$\frac{1}{2}\frac{d}{d\tau}(V_1^2 - V_2^2) = -\lambda\,\beta(\tau)(\alpha_2 |V_2|^{q+1} - \alpha_1 |V_1|^{p+1}) \sim -\lambda\,\beta(\xi)(\alpha_2 - \alpha_1)(2\,|\kappa|)^{\frac{q+1}{2}}\,(\xi - \tau)^{\frac{q+1}{2}} < 0.$$

Finally, an integration of the previous equation gives $V_1^2(\tau) - V_2^2(\tau) > 0$ for $\tau \in (\xi - \epsilon, \xi)$ for small $\epsilon > 0$ which is in contradiction with (3.3).

Note that the function $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$ is actually *strictly* monotone in any interval of the form $[0, \alpha_1]$ –provided we are in the regime that $\hat{\tau}_1(\tau_0, \alpha_1) > \tau_3$.

Now, we have to prove the continuity of $\hat{\tau}_1(\tau_0, \alpha)$, $\check{\tau}_+(\tau_2, \alpha)$ with respect to α and (3.2). To this end, let us introduce the following notations:

Given $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$, we set $W_j = W_{\alpha_j} = V_{\alpha_j}^2 = V_j^2$, $\delta = \frac{q+1}{2}$ and we rewrite (2.6) in the form

$$\frac{dW_j}{d\tau}(\tau) + 2\,\alpha_j\,\lambda\,\beta(\tau) W_j^\delta(\tau) = 2\,G_\tau(\tau_0, \lambda, \tau), \quad \text{if} \quad V_j > 0, \tag{3.4}$$

and

$$\frac{dW_j}{d\tau}(\tau) - 2\,\alpha_j\,\lambda\,\beta(\tau) W_j^\delta(\tau) = 2\,G_\tau(\tau_0, \lambda, \tau), \quad \text{if} \quad V_j < 0. \tag{3.5}$$

Proof of the continuity of $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$**.**

Assume that $0 \le \alpha_1 < \alpha_2 \le \overline{\alpha}_0$ and V_1 and V_2 represent here the trajectories $C^1_{\tau_0, \alpha_1}$ and respectively $C^1_{\tau_0, \alpha_2}$. In the special case $\alpha = \overline{\alpha}_0$, $V_{\overline{\alpha}_0}$ will represent both $C^1_{\tau_0, \overline{\alpha}_0}$ and $C^1_{\tau_2, \overline{\alpha}_0}$. Then in one hand, thanks to Proposition 3.5,

$$W_1(\tau) \ge W_2(\tau) \quad \text{for} \quad \tau \in [\tau_0, \hat{\tau}_+(\tau_2, \overline{\alpha}_0)]. \tag{3.6}$$

On the other hand, using (3.4) for $j = 1, 2$ we get after an integration over $[\tau_0, \tau]$

$$W_1(\tau) - W_2(\tau) = 2\,(\alpha_2 - \alpha_1)\lambda \int_{\tau_0}^{\tau} \beta(s)\, W_1^\delta(s) ds + 2\,\alpha_2\,\lambda \int_{\tau_0}^{\tau} \beta(s)\,(W_2^\delta(s) - W_1^\delta(s))\, ds. \tag{3.7}$$

Now, combining (3.6) and (3.7) we obtain

$$0 \le W_1(\tau) - W_2(\tau) \le C(\alpha_2 - \alpha_1). \tag{3.8}$$

The last inequality gives the continuity of $\alpha \mapsto V_\alpha(\tau)$ on the interval $[0, \overline{\alpha}_0]$, for all $\tau \in [\tau_0, \hat{\tau}_+(\tau_2, \overline{\alpha}_0)]$. In particular, we have

$$\lim_{\alpha \to \overline{\alpha}_0} V_\alpha(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) = V_{\overline{\alpha}_0}(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) = 0.$$

Let us prove now the continuity of $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$. First, note that for all $0 \le \alpha_1 < \alpha_2 \le \overline{\alpha}_0$, (3.8) remains valid for all $\tau \in [\tau_0, \hat{\tau}_1(\tau_0, \alpha_2)]$. Assume now that for a fixed $\alpha_1 < \overline{\alpha}_0$, $V_1(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) > 0$. Then by the continuity of $\alpha \mapsto V_\alpha(\hat{\tau}_+(\tau_2, \overline{\alpha}_0))$ proved before, there is $\alpha_1 < \tilde{\alpha} < \overline{\alpha}_0$ such that for all $\alpha \in [0, \tilde{\alpha}]$, $V_\alpha(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) > 0$. Then, taking α_2 such that $\alpha_1 < \alpha_2 < \tilde{\alpha}$, and thanks to the monotony of $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$,

$$\tau_3 \le \hat{\tau}_+(\tau_2, \overline{\alpha}_0) < \hat{\tau}_1(\tau_0, \tilde{\alpha}) \le \hat{\tau}_1(\tau_0, \alpha_2) \le \hat{\tau}_1(\tau_0, \alpha_1). \tag{3.9}$$

Note that for $\tau \ge \hat{\tau}_1(\tau_0, \tilde{\alpha})$, since $G_{\tau\tau} = -p' - \lambda^2 < 0$, we have,

$$G_\tau(\tau_0, \lambda, \tau) \le G_\tau(\tau_0, \lambda, \hat{\tau}_1(\tau_0, \tilde{\alpha})) = \kappa < 0. \tag{3.10}$$

Integrating (3.4) for $j = 1$, over $[\hat{\tau}_1(\tau_0, \alpha_2), \hat{\tau}_1(\tau_0, \alpha_1)]$, we get

$$W_1(\hat{\tau}_1(\tau_0, \alpha_2)) \ge 2\,|\kappa|(\hat{\tau}_1(\tau_0, \alpha_1) - \hat{\tau}_1(\tau_0, \alpha_2)). \tag{3.11}$$

Now, injecting (3.8) for $\tau = \hat{\tau}_1(\tau_0, \alpha_2)$ in (3.11) we get

$$2\,|\kappa|(\hat{\tau}_1(\tau_0, \alpha_1) - \hat{\tau}_1(\tau_0, \alpha_2)) \le C(\alpha_2 - \alpha_1). \tag{3.12}$$

A similar inequality is obtained when $\alpha_2 < \alpha_1$, by exchanging the role of α_1 and α_2. Thus, the continuity of $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$ at $\alpha = \alpha_1$ when $V_1(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) > 0$ is achieved.

Consider now the eventual case that is $V_1(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) = 0$. Then, since $\alpha \mapsto \hat{\tau}_1(\tau_0, \alpha)$ is monotone decreasing and the monotony is strict in a region strictly at the right hand side of τ_3 we have necessarily $\hat{\tau}_+(\tau_2, \overline{\alpha}_0) = \tau_3$ and,

$$\hat{\tau}_1(\tau_0, \alpha) = \hat{\tau}_1(\tau_0, \alpha_1) = \hat{\tau}_+(\tau_2, \overline{\alpha}_0) = \tau_3, \quad \forall \quad \alpha_1 \le \alpha \le \overline{\alpha}_0. \tag{3.13}$$

Now, if $\alpha_2 < \alpha_1$, since $G_{\tau\tau\tau} = -p'' < 0$ for $\tau \ge \tau_3 > c$, an integration of (3.4) over $[\tau_3, \hat{\tau}_1(\tau_0, \alpha_2)]$, for $j = 2$ gives

$$|G_{\tau\tau}(\tau_0, \lambda, \tau_3)|(\hat{\tau}_1(\tau_0, \alpha_2) - \hat{\tau}_1(\tau_0, \alpha_1))^2 \le W_2(\tau_3) = W_2(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) - W_1(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)). \tag{3.14}$$

Injecting (3.8) for $\tau = \hat{\tau}_+(\tau_2, \overline{\alpha}_0)$ in (3.14) we obtain

$$|G_{\tau\tau}(\tau_0, \lambda, \tau_3)|(\hat{\tau}_1(\tau_0, \alpha_2) - \hat{\tau}_1(\tau_0, \alpha_1))^2 \le C(\alpha_1 - \alpha_2). \tag{3.15}$$

Since $G_{\tau\tau}(\tau_0, \lambda, \tau_3) \ne 0$, we deduce the continuity at $\alpha = \alpha_1$ when $V_{\alpha_1}(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) = 0$.

Now, consider the limit case that is $\alpha_1 = \overline{\alpha}_0$. We have $V_{\overline{\alpha}_0}(\hat{\tau}_+(\tau_2, \overline{\alpha}_0)) = 0$ and then, either $\hat{\tau}_+(\tau_2, \overline{\alpha}_0) > \tau_3$, then we have an estimate like (3.12) that is

$$2\,|\kappa|(\hat{\tau}_1(\tau_0, \alpha_2) - \hat{\tau}_+(\tau_2, \overline{\alpha}_0)) \le C\,(\overline{\alpha}_0 - \alpha_2),$$

or $\hat{\tau}_+(\tau_2, \overline{\alpha}_0) = \tau_3$ and then (3.15) remains valid. We obtain in both cases that

$$\lim_{\alpha \to \overline{\alpha}_0} \hat{\tau}_1(\tau_0, \alpha) = \hat{\tau}_+(\tau_2, \overline{\alpha}_0).$$

Proof of the continuity of $\alpha \mapsto \check{\tau}_+(\tau_2, \alpha)$. Contrarily to the function $\hat{\tau}_1$, the last quantity does not exist necessarily for large values of α and for $q > 1$. So let us prove that $\check{\tau}_+(\tau_2, \alpha)$ exists and is continuous in some interval $[0, \alpha_{max})$ where $\alpha_{max} \le +\infty$, and depends a priori on τ_2 and q.

So assume that $\check{\tau}_+(\tau_2, \alpha_1)$ exists for some value $\alpha_1 \ge 0$. Then, since $\check{\tau}_+(\tau_2, 0)$ always exists (for all $q \ge 0$), and thanks to the monotonicity of $C^4_{\tau_2,\alpha}$, $\check{\tau}_+(\tau_2, \alpha)$ exists for all $\alpha \in [0, \alpha_1]$. Now to prove that $\check{\tau}_+(\tau_2, \alpha_2)$ also exists for α_2 in a right neighborhood of α_1, we have to prove first that for $\alpha_2 \in [\alpha_1, \alpha_1 + \eta]$ where η is sufficiently small, $W_2(\tau)$ exists for all $\tau \in I_1 = [\tau_2, \check{\tau}_+(\tau_2, \alpha_1)]$. To this end we introduce the well-known Gronwall Lemma:

Lemma 3.6. *Let φ and ψ be two continuous, non-negative functions defined on an interval $I = [t_1, t_2]$ such that, for all $t \in I$,*

$$\phi(t) \le c_1 \int_{t_1}^{t} \psi(s)\,\phi(s)\,ds + c_2,$$

where $c_1, c_2 > 0$. *Then, for all* $t \in I$

$$\phi(t) \le c_2\, e^{c_1 \int_{t_1}^{t} \psi(s)\, ds}.$$

Given $\alpha_2 > \alpha_1$ we distinguish between two cases:
Case (a) : If $\delta < 1$, then setting $A(\tau) = \sup_{s\in[\tau_2,\tau]} W_2(s)$, an integration of (3.5) over the interval $[\tau_2, \tau]$, where $\tau \in I_1 = [\tau_2, \check{\tau}_+(\tau_2, \alpha_1)]$, gives

$$A(\tau) \le C A(\tau)^{\delta} + C'.$$

The last inequality guaranteed to us that $W_2(\tau)$ exists for all $\alpha \ge \alpha_1$ and $\tau \in I_1$.
Case (b) : If $\delta \ge 1$, then while $W_2(\tau)$ exists, integrating (3.5) for $j = 1, 2$ over the interval $[\tau_2, \tau]$, $\tau \in I_1$, we get

$$|W_2(\tau) - W_1(\tau)| \le 2\,(\alpha_2 - \alpha_1)\lambda \int_{\tau_2}^{\tau} \beta(s)\, W_1^{\delta}(s) ds + 2\,\alpha_2\,\lambda \int_{\tau_2}^{\tau} \beta(s)\, |W_2^{\delta}(s) - W_1^{\delta}(s)|\, ds$$

$$\le K(\alpha_2 - \alpha_1) + C \int_{\tau_0}^{\tau} W_2(s)^{\delta-1} |W_2(s) - W_1(s)|\, ds. \tag{3.16}$$

Then, applying Lemma3.6, we obtain an estimate of the form

$$|W_2(\tau) - W_1(\tau)| \le K(\alpha_2 - \alpha_1) e^{C \int_{\tau_2}^{\tau} W_2(s)^{\delta-1}\, ds}. \tag{3.17}$$

Then, setting

$$M = \sup_{\tau\in I_1} |W_1(\tau)| \quad \text{and} \quad M' = 2M, \tag{3.18}$$

there is $\eta > 0$ such that,

$$M + K\,\eta\, e^{C\, M'^{\delta-1} |I_1|} < M'. \tag{3.19}$$

Then, it is easy to deduce from (3.17)..(3.19) that for all $\alpha_2 \in [\alpha_1, \alpha_1 + \eta]$, and $\tau \in I_1$, $W_2(\tau)$ exists and $|W_2(\tau)| < M'$.

Let us prove now the continuity of $\alpha \mapsto \check{\tau}_+(\tau_2, \alpha)$ at the value $\alpha = \alpha_1$. To this end we set $\tilde{\alpha}_\pm = \alpha_1 \pm \eta$ and we introduce the reals $\tilde{\tau} = \frac{\tau_3 + \overline{\tau}}{2}$, $\overline{C}$ and $\overline{C}_1$ such that

$$\overline{\alpha}_0\, \lambda\, (\sup_{\tau\in[\tau_3,\tilde{\tau}_+]} |\beta(\tau)|) \overline{C}^{\delta} = -\frac{G_\tau(\tilde{\tau})}{2} \quad \text{and} \quad \overline{C}_1 = min(\overline{C}, W_0(\tilde{\tau})), \tag{3.20}$$

where $\tilde{\tau}_+ \ge \check{\tau}_+(\tau_2, \alpha_1)$ will be chosen later. Then by (3.5) and since $G_{\tau\tau} = -p' - \lambda^2 < 0$,

$$\forall\, \alpha_2 \in [0, \overline{\alpha}_0], \forall\, \tau \in [\tilde{\tau}, \tilde{\tau}_+], \quad \text{if} \quad W_2(\tau) \le \overline{C}_1 \;\; \text{then} \;\; \frac{dW_2}{d\tau}(\tau) \le G_\tau(\tilde{\tau}) < 0. \tag{3.21}$$

On the other hand, if $\alpha_2 \in [\tilde{\alpha}_-, \alpha_1]$, since $W_2(\tilde{\tau}) > W_0(\tilde{\tau}) \ge \overline{C}_1$ and $W_2(\check{\tau}_+(\tau_2, \alpha_2)) = 0$, there exists a first value $\xi \in (\tilde{\tau}, \check{\tau}_+(\tau_2, \alpha_2))$ such that $W_2(\xi) = \overline{C}_1$ and $W_2(\tau) < \overline{C}_1$ for $\tau \in [\xi, \check{\tau}_+(\tau_2, \alpha_2)]$. Thus necessarily,

$$\forall\, \tau \in (\tilde{\tau}, \xi), \quad W_2(\tau) > \overline{C}_1. \tag{3.22}$$

Otherwise, there will exist $\xi' \in (\tilde{\tau}, \xi)$ such that $W_2(\xi') = \overline{C}_1$ and $\frac{dW_2}{d\tau}(\xi') \ge 0$ which is in contradiction with (3.21). Also, (3.21) gives

$$W_2(\tau) \ge |G_\tau(\tilde{\tau})| (\check{\tau}_+(\tau_2, \alpha_2) - \tau) \quad \forall\, \tau \in [\xi, \check{\tau}_+(\tau_2, \alpha_2)]. \tag{3.23}$$

On the other hand, using the asymptotic properties of W_0 near τ_2 and since $W_0(\tau) > 0$ in the interval $(\tau_2, \tilde{\tau}]$, we obtain

$$W_2(\tau) \ge W_0(\tau) \ge C_0(\tau - \tau_2)^2 \quad \text{if} \quad \tau \in [\tau_2, \tilde{\tau}] \tag{3.24}$$

Finally, using (3.22), (3.23) and (3.24) we obtain that for $1/2 < \delta < 1$, (i.e. $0 < q < 1$), there exists $C_1 > 0$ such that

$$\forall\, \alpha_2 \in [\tilde\alpha_-, \alpha_1], \qquad \int_{\tau_2}^{\check\tau_+(\tau_2,\alpha_2)} W_2(s)^{\delta-1} \le C_1. \tag{3.25}$$

For $\alpha_2 \in [\alpha_1-\eta, \alpha_1+\eta] = [\tilde\alpha_-, \tilde\alpha_+]$, let us integrate (3.5) over the interval $[\tau_2, \tau]$ where $\tau \in I_2 = [\tau_2, \check\tau_+(\tau_2,\alpha_2)]$ if $\alpha_2 < \alpha_1$ and $\tau \in I_1 = [\tau_2, \check\tau_+(\tau_2,\alpha_1)]$ if $\alpha_2 > \alpha_1$. We obtain

$$|W_2(\tau) - W_1(\tau)| \le 2\,|\alpha_2-\alpha_1|\lambda \int_{\tau_0}^{\tau} \beta(s)\, W_1^{\delta}(s) ds + 2\,\alpha_2\,\lambda \int_{\tau_0}^{\tau} \beta(s)\, |W_2^{\delta}(s) - W_1^{\delta}(s)|\, ds$$

$$\le K|\alpha_2-\alpha_1| + C\int_{\tau_0}^{\tau} \psi(s)|W_2(s)-W_1(s)|\, ds, \tag{3.26}$$

where,

$$\psi = \begin{cases} W_{\tilde\alpha_+}^{\delta-1} & \text{if} \quad \delta>1, \\ W_2^{\delta-1} & \text{if} \quad 1/2 \le \delta < 1 \quad \text{and} \quad \alpha_2 < \alpha_1, \\ W_1^{\delta-1} & \text{if} \quad 1/2 \le \delta < 1 \quad \text{and} \quad \alpha_2 > \alpha_1, \\ 1 & \text{if} \quad \delta = 1. \end{cases} \tag{3.27}$$

Here $s \mapsto W_{\tilde\alpha_+}(s) = V_{\tilde\alpha_+}^2(s)$ is the curve corresponding to $C^4_{\tau_2,\tilde\alpha_+}$. Then, applying Lemma 3.6 we obtain from (3.26)

$$|W_2(\tau) - W_1(\tau)| \le K(\alpha_2-\alpha_1)e^{C\int_{\tau_0}^{\tau}\psi(s)ds}. \tag{3.28}$$

Thanks to (3.25), (3.27) and (3.28), and for $\delta > 1/2$ (i.e. $q>0$) we obtain

$$|W_2(\tau) - W_1(\tau)| \le K'|\alpha_2-\alpha_1|, \qquad \text{for} \quad \tau \in [\tau_2,\, \min(\check\tau_+(\tau_2,\alpha_1), \check\tau_+(\tau_2,\alpha_2))], \tag{3.29}$$

where K' is a constant independent of $\alpha_2 \in [\tilde\alpha_-, \tilde\alpha_+]$.

Consider now the more complicated case that is $q=0$ (i.e. $\delta = 1/2$). Thanks to the asymptotic properties of the trajectories near τ_2 there exists $\tilde C > 0$ such that

$$W_{\tilde\alpha_+}(\tau) \le \tilde C(\tau-\tau_2)^2 \quad \textit{for} \quad \tau \in [\tau_2, \tilde\tau]. \tag{3.30}$$

On the other hand, since $\lim_{\alpha\to\alpha_1} \mu_+(\tau_2,\lambda,\alpha) = \mu_+(\tau_2,\lambda,\alpha_1)$, and by (2.3), $\mu_+(\tau_2,\lambda,\alpha_1) < -\alpha_1\,\lambda\,\beta(\tau_2)$, we can choose η sufficiently small such that $\tilde\alpha_- = \alpha_1-\eta$ satisfies

$$\mu_+(\tau_2,\lambda,\tilde\alpha_-) \le -\alpha_1\lambda\,\beta(\tau_2). \tag{3.31}$$

Now, once $\tilde\alpha_-$ fixed such that (3.31), we have

$$\lim_{\tau\to\tau_2} \frac{\beta(\tau)\,(\tau-\tau_2)}{2\,\sqrt{W_{\tilde\alpha_-}(\tau)}} = \frac{\beta(\tau_2)}{2\,|\mu_+(\tau_2,\lambda,\tilde\alpha_-)|}. \tag{3.32}$$

Then, thanks to (3.31) and (3.32), there is $\zeta \in (\tau_2, \tilde\tau)$ such that

$$\forall \tau \in [\tau_2, \zeta], \quad \frac{\beta(\tau)\,(\tau-\tau_2)}{2\,\sqrt{W_{\tilde\alpha_-}(\tau)}} \le \frac{3}{4\,\alpha_1\,\lambda}. \tag{3.33}$$

Let us introduce a real $\epsilon > 0$ sufficiently small such that we have $\tilde\zeta = \tau_2 + \epsilon < \zeta$.

Then by monotony and using (3.30) we get

$$\forall \tau \in [\tau_2, \tilde\zeta], \quad |W_2(\tau) - W_1(\tau)| \le W_{\tilde\alpha_+}(\tau) \le \tilde C\epsilon^2. \tag{3.34}$$

Now, if $\tau \in [\tilde\zeta, \min(\check\tau_+(\tau_2,\alpha_2), \check\tau_+(\tau_2,\alpha_2))]$, then (3.26) is replaced by

$$|W_2(\tau) - W_1(\tau)| \le |W_2(\tilde\zeta) - W_1(\tilde\zeta)| + K|\alpha_2-\alpha_1| + 2\,\alpha_1\,\lambda \textstyle\int_{\tilde\zeta}^{\tau} \beta(s)|\sqrt{W_2(s)} - \sqrt{W_1(s)}|\, ds$$

$$\le \tilde C\,\epsilon^2 + K|\alpha_2-\alpha_1| + 2\,\alpha_1\,\lambda \int_{\tilde\zeta}^{\tau} \beta(s)\phi(s)|W_2(s)-W_1(s)|\, ds, \tag{3.35}$$

where

$$\phi(s) = \begin{cases} \frac{1}{2\sqrt{W_{\tilde{\alpha}_-}(s)}} & \text{for} \quad s \in [\tilde{\zeta}, \zeta], \\ \frac{1}{2\sqrt{W_0(s)}} & \text{for} \quad s \in [\zeta, \tilde{\tau}], \\ \frac{1}{2\sqrt{W_2(s)}} & \text{for} \quad s \in [\tilde{\tau}, \check{\tau}_+(\tau_2, \alpha_2)] \quad \text{if} \quad \alpha_2 < \alpha_1, \\ \frac{1}{2\sqrt{W_1(s)}} & \text{for} \quad s \in [\tilde{\tau}, \check{\tau}_+(\tau_2, \alpha_1)] \quad \text{if} \quad \alpha_2 > \alpha_1. \end{cases} \tag{3.36}$$

Thanks to (3.35) and Lemma 3.6 we get

$$|W_2(\tau) - W_1(\tau)| \le (\tilde{C}\,\epsilon^2 + K|\alpha_2 - \alpha_1|)e^{2\,\alpha_1\,\lambda\,\int_{\tilde{\zeta}}^{\tau} \beta(s)\,\phi(s)\,ds}. \tag{3.37}$$

Writing

$$\int_{\tilde{\zeta}}^{\tau} \beta(s)\,\phi(s)\,ds = \int_{\tilde{\zeta}}^{\zeta} \beta(s)\,\phi(s)\,ds + \int_{\zeta}^{\tilde{\tau}} \beta(s)\,\phi(s)\,ds + \int_{\tilde{\tau}}^{\tau} \beta(s)\,\phi(s)\,ds,$$

then, in one hand, using (3.33) and (3.36) we obtain

$$2\,\alpha_1\,\lambda \int_{\tilde{\zeta}}^{\zeta} \beta(s)\,\phi(s)\,ds \le \frac{3}{2}\int_{\tilde{\zeta}}^{\zeta} \frac{1}{s - \tau_2} ds \le C - \frac{3}{2}\ln(\epsilon). \tag{3.38}$$

On the other hand, using (3.22) and (3.23) we get

$$\alpha_1\,\lambda \int_{\tilde{\tau}}^{\tau} \beta(s)\,\phi(s)\,ds \le C. \tag{3.39}$$

Finally, injecting (3.38) and (3.39) in (3.37) we obtain that for all $\tau \in [\tau_2, \min(\check{\tau}_+(\tau_2, \alpha_2), \check{\tau}_+(\tau_2, \alpha_1))]$,

$$|W_2(\tau) - W_1(\tau)| \le (\tilde{C}\epsilon^2 + K|\alpha_2 - \alpha_1|)\frac{C'}{\epsilon^{3/2}} \le C''\sqrt{\epsilon} + \frac{K'}{\epsilon^{3/2}}|\alpha_2 - \alpha_1|. \tag{3.40}$$

Thanks to (3.29) and (3.40) we will conclude the prove of continuity in the two situations $\delta > 1/2$ and respectively $\delta = 1/2$ as follows:

If $\delta > 1/2$, choosing η sufficiently small such that thanks to (3.29) and (3.21) we have for $\alpha_2 \in [\tilde{\alpha}_-, \alpha_1]$

$$|G_\tau(\tilde{\tau})|(\check{\tau}_+(\tau_2, \alpha_1) - \check{\tau}_+(\tau_2, \alpha_2)) \le W_1(\check{\tau}_+(\tau_2, \alpha_2)) \le K'|\alpha_2 - \alpha_1| \le \overline{C}_1 \tag{3.41}$$

If $\delta = 1/2$, choosing ϵ small enough and then η sufficiently small, we get similarly from (3.40) and (3.21) that

for $\alpha_2 \in [\tilde{\alpha}_-, \alpha_1]$

$$|G_\tau(\tilde{\tau})|(\check{\tau}_+(\tau_2, \alpha_1) - \check{\tau}_+(\tau_2, \alpha_2)) \le W_1(\check{\tau}_+(\tau_2, \alpha_2)) \le C''\sqrt{\epsilon} + \frac{K'}{\epsilon^{3/2}}|\alpha_2 - \alpha_1| \le \overline{C}_1 \tag{3.42}$$

Inequalities (3.41) and (3.42) give the left continuity of $\alpha \mapsto \check{\tau}_+(\tau_2, \alpha)$ at $\alpha = \alpha_1$ for all $\delta \ge 1/2$.

Let us conclude now the right continuity. Choosing η (and ϵ if $\delta = 1/2$) sufficiently small we obtain from (3.41) and (3.42) that for $\alpha_2 \in [\alpha_1, \tilde{\alpha}_+]$, and thus by (3.21), $W_2(\check{\tau}_+(\tau_2, \alpha_1)) \le \overline{C}_1$ and thus $\check{\tau}_+(\tau_2, \alpha_2)$ exists and

$$|G_\tau(\tilde{\tau})|(\check{\tau}_+(\tau_2, \alpha_2) - \check{\tau}_+(\tau_2, \alpha_1)) \le W_2(\check{\tau}_+(\tau_2, \alpha_1)) \le K'|\alpha_2 - \alpha_1| \le \overline{C}_1, \tag{3.43}$$

for $\delta > 1/2$ and,

$$|G_\tau(\tilde{\tau})|(\check{\tau}_+(\tau_2, \alpha_2) - \check{\tau}_+(\tau_2, \alpha_1)) \le W_2(\check{\tau}_+(\tau_2, \alpha_1)) \le C''\sqrt{\epsilon} + \frac{K'}{\epsilon^{3/2}}|\alpha_2 - \alpha_1| \le \overline{C}_1, \tag{3.44}$$

for $\delta = 1/2$. Inequalities (3.43) and (3.44) give us the right continuity of $\alpha \mapsto \check{\tau}_+(\tau_2, \alpha)$ at $\alpha = \alpha_1$. □

Proof of Theorem 3.4. One sees that there exists $\overline{\alpha}_1 = \overline{\alpha}_1(\tau_0, \lambda)$ for which there is a trajectory with one oscillation connecting τ_0 to τ_2 if and only if

$$\hat{\tau}_1(\tau_0, \overline{\alpha}_1) = \check{\tau}_+(\tau_2, \overline{\alpha}_1).$$

So let us prove the existence of such value. We have $\check{\tau}_+(\tau_2, 0) = \hat{\tau}_+(\tau_2, 0) = \overline{\tau}$, where $\overline{\tau}$ is defined in Proposition 2.3, and we may distinguish between two cases related to the parameter α_{max} defined in Proposition3.5.

Case (a) : $\alpha_{max} \leq \overline{\alpha}_0$. Then $\check{\tau}_+(\tau_2, \overline{\alpha}_0)$ does not exist but there is $0 < \tilde{\alpha}_0 < \alpha_{max} \leq \overline{\alpha}_0$ such that $\check{\tau}_+(\tau_2, \tilde{\alpha}) = \hat{\tau}_1(\tau_0, 0) = \tau_{max}$. Thus, the function

$$\alpha \mapsto \tilde{\tau}_1(\alpha) = \check{\tau}_+(\tau_2, \alpha) - \hat{\tau}_1(\tau_0, \alpha)$$

is well defined on the interval $[0, \tilde{\alpha}_0]$, continuous and strictly monotone increasing (by Proposition 3.5), and satisfies in one hand,

$$\begin{aligned}\tilde{\tau}_1(0) &= \check{\tau}_+(\tau_2, 0) - \hat{\tau}_1(\tau_0, 0)\\ &= \hat{\tau}_+(\tau_2, 0) - \hat{\tau}_1(\tau_0, 0)\\ &= \overline{\tau} - \tau_{max} < 0,\end{aligned}$$

and in the other hand,

$$\begin{aligned}\tilde{\tau}_1(\tilde{\alpha}_0) &= \check{\tau}_+(\tau_2, \tilde{\alpha}_0) - \hat{\tau}_1(\tau_0, \tilde{\alpha}_0)\\ &= \hat{\tau}_1(\tau_0, 0) - \hat{\tau}_1(\tau_0, \tilde{\alpha}_0) \geq 0.\end{aligned}$$

The existence and uniqueness of $\overline{\alpha}_1$ is thus clear.

Case (b) : $\alpha_{max} > \overline{\alpha}_0$. Then we consider the same function $\alpha \mapsto \tilde{\tau}_1(\alpha)$ on the whole interval $[0, \overline{\alpha}_0)$. We have as above $\tilde{\tau}_1(0) < 0$ and by (3.2)

$$\lim_{\alpha \to \overline{\alpha}_0} \tilde{\tau}_1(\alpha) = \check{\tau}_+(\tau_2, \overline{\alpha}_0) - \lim_{\alpha \to \overline{\alpha}_0} \hat{\tau}_1(\tau_0, \alpha) = \check{\tau}_+(\tau_2, \overline{\alpha}_0) - \hat{\tau}_+(\tau_2, \overline{\alpha}_0).$$

But, since

$$\check{\tau}_+(\tau_2, \overline{\alpha}_0) > \check{\tau}_+(\tau_2, 0) = \overline{\tau} = \hat{\tau}_+(\tau_2, 0) \geq \hat{\tau}_+(\tau_2, \overline{\alpha}_0),$$

we obtain that $\lim_{\alpha \to \overline{\alpha}_0} \tilde{\tau}_1(\alpha) > 0$. With the same arguments as the ones in Case (a), the existence and uniqueness of $\overline{\alpha}_1(\tau_0, \lambda)$ follows.

For the values $\alpha \in (\overline{\alpha}_1, \overline{\alpha}_0)$, it is easy to see that $\mathcal{C}^1_{\tau_0,\alpha}$ remains into the domain delimited by $\mathcal{C}^1_{\tau_0,\overline{\alpha}_0} \cup \mathcal{C}^1_{\tau_0,\overline{\alpha}_1}$ and necessarily converges to the equilibrium point $(\tau_3, 0)$.

4. Traveling waves with two or more oscillations

We now turn to trajectories with multiple changes of monotonicity:

Theorem 4.1. *There exists a strictly decreasing sequence* $(\overline{\alpha}_n(\tau_0, \lambda))$ *of values of the parameter* α *such that there is a traveling wave with* n *oscillations connecting* τ_0 *to* τ_2 *if and only if* $\alpha = \overline{\alpha}_n$. *Moreover,*

– *for* $\alpha \in (\overline{\alpha}_{2n+2}, \overline{\alpha}_{2n+1})$ *or* $\alpha > \overline{\alpha}_0$ *there exists a connection from* τ_0 *to* τ_1.
– *for* $\alpha \in (\overline{\alpha}_{2n+1}, \overline{\alpha}_{2n})$ *there exists a connection from* τ_0 *to* τ_3.

In addition, the sequence $\overline{\alpha}_n$ *tends to zero:*

$$\lim_n \overline{\alpha}_n(\tau_0, \lambda) = 0. \tag{4.1}$$

To prove the existence of $\overline{\alpha}_2$, we need the following:

Proposition 4.2.

• *The function* $\alpha \mapsto \hat{\tau}_2(\tau_0, \alpha)$ *is continuous and monotone increasing on the interval* $[0, \overline{\alpha}_1)$ *and satisfies*

$$\lim_{\alpha \to \overline{\alpha}_1} \hat{\tau}_2(\tau_0, \alpha) = \hat{\tau}_-(\tau_2, \overline{\alpha}_1) \tag{4.2}$$

• *The function* $\alpha \mapsto \check{\tau}_-(\tau_2, \alpha)$ *is continuous and* strictly *monotone decreasing on* $[0, \overline{\alpha}_0]$.

Proof. First, the monotony of $\hat{\tau}_2(\tau_0,\alpha)$ and $\check{\tau}_-(\tau_2,\alpha)$ with respect to α is a consequence of Proposition 2.2. Now, given two reals $0 \le \alpha_1 < \alpha_2 < \overline{\alpha}_1$, we set $W_j = W_{\alpha_j}$ for $j=1,2$, the corresponding trajectories of $C^1_{\tau_0,\alpha_j}$ in the half space $V \le 0$. Then, from (3.5) we have

$$\frac{dW_1}{d\tau} > 2\,G_\tau(\tau_0,\lambda,\tau) > -C,$$

where the constant $C > 0$ is independent of α. The integration of the last inequality over the interval $[\hat{\tau}_1(\tau_0,\alpha_2),\hat{\tau}_1(\tau_0,\alpha_1)]$ gives

$$|W_1(\hat{\tau}_1(\tau_0,\alpha_2)) - W_2(\hat{\tau}_1(\tau_0,\alpha_2))| = W_1(\hat{\tau}_1(\tau_0,\alpha_2)) \le C(\hat{\tau}_1(\tau_0,\alpha_1) - \hat{\tau}_1(\tau_0,\alpha_2)). \tag{4.3}$$

Up to add an estimate like (4.3), the proofs of the continuity of $\alpha \mapsto \hat{\tau}_2(\tau_0,\alpha)$ on the interval of the form $[\delta,\overline{\alpha}_1)$, where $\delta > 0$, and (4.2) are obtained similarly as the first item of Proposition3.5.

Now, for $\alpha_2 > 0$ in the neighborhood of $\alpha_1 = 0$, it is easy to obtain from (3.5) and (4.3) the estimate

$$|W_2(\tau) - W_0(\tau)| \le C'\,\alpha_2 + C(\hat{\tau}_1(\tau_0,0) - \hat{\tau}_1(\tau_0,\alpha_2)), \quad \forall\,\tau \in [\hat{\tau}_2(\tau_0,\alpha_2),\hat{\tau}_1(\tau_0,\alpha_2)], \tag{4.4}$$

where $C' > 0$ is independent of α. On the other hand, $W_0(\tau)$ satisfies an estimate of the form:

$$W_0(\tau) \ge C_0(\tau-\tau_0)^2 \quad \text{for } \tau \in [\tau_0,\tau_1], \tag{4.5}$$

which combined with (4.4), applied at the point $\tau = \hat{\tau}_2(\tau_0,\alpha_2)$ gives

$$C_0(\hat{\tau}_2(\tau_0,\alpha_2) - \tau_0)^2 \le C'\,\alpha_2 + C(\hat{\tau}_1(\tau_0,0) - \hat{\tau}_1(\tau_0,\alpha_2)). \tag{4.6}$$

Finally, by (4.6), thanks to the continuity of $\alpha \mapsto \hat{\tau}_1(\tau_0,\alpha)$ at $\alpha = 0$ we get the continuity of $\alpha \mapsto \hat{\tau}_2(\tau_0,\alpha)$ at the same point. Thus, we conclude the continuity of $\alpha \mapsto \hat{\tau}_2(\tau_0,\alpha)$ in the whole interval $[0,\overline{\alpha}_1)$.

The second item in Proposition 4.2 seems to be a symmetric case to the second item in Proposition 3.5, but with less difficulties since there is no problem of existence of $\check{\tau}_-(\tau_2,\alpha)$ for $\alpha \in [0,\overline{\alpha}_0)$. This is due to the specific monotonicity of $\alpha \mapsto C^2_{\tau_2,\alpha}$.

□

Proof of Theorem 4.1. In the same way as for $\overline{\alpha}_0$, it is clear that there exists $\overline{\alpha}_2 = \overline{\alpha}_2(\tau_0,\lambda)$ for which there is a connection with two oscillations connecting τ_0 and τ_2, iff

$$\hat{\tau}_2(\tau_0,\overline{\alpha}_2) = \check{\tau}_-(\tau_2,\overline{\alpha}_2).$$

$\overline{\alpha}_2$ is obtained similarly as $\overline{\alpha}_1$, by considering the function

$$\alpha \mapsto \tilde{\tau}_2(\alpha) = \check{\tau}_-(\tau_2,\alpha) - \hat{\tau}_2(\tau_0,\alpha),$$

on the whole interval $[0,\overline{\alpha}_1)$. In fact,by Proposition 4.2, function $\tilde{\tau}_2$ is strictly monotone decreasing and satisfies in one hand

$$\tilde{\tau}_2(0) = \underline{\tau} - \tau_0 > 0. \tag{4.7}$$

On the other hand, using (4.2) we have

$$\lim_{\alpha\to\overline{\alpha}_1} \tilde{\tau}_2(\alpha) = \check{\tau}_-(\tau_2,\overline{\alpha}_1) - \lim_{\alpha\to\overline{\alpha}_1} \hat{\tau}_2(\tau_0,\alpha) = \check{\tau}_-(\tau_2,\overline{\alpha}_1) - \hat{\tau}_-(\tau_2,\overline{\alpha}_1). \tag{4.8}$$

But, by monotonicity of $\alpha \mapsto C^2_{\tau_2,\alpha}$ and $\alpha \mapsto C^3_{\tau_2,\alpha}$ we get

$$\check{\tau}_-(\tau_2,\overline{\alpha}_1) \le \underline{\tau} = \check{\tau}_-(\tau_2,0) = \hat{\tau}_-(\tau_2,0) \le \hat{\tau}_-(\tau_2,\overline{\alpha}_1). \tag{4.9}$$

Then, injecting (4.9) in (4.8) we get $\lim_{\alpha\to\overline{\alpha}_1} \tilde{\tau}_2(\alpha) \le 0$, which combined with (4.7) give the existence and uniqueness of $\overline{\alpha}_2(\tau_0,\lambda)$.

Now, for $\alpha \in (\overline{\alpha}_2,\overline{\alpha}_1)$, $\mathcal{C}^1_{\tau_0,\alpha}$ remains into the domain delimited by $\mathcal{C}^1_{\tau_0,\overline{\alpha}_1} \cup \mathcal{C}^1_{\tau_0,\overline{\alpha}_2}$ and necessarily converges to $(\tau_1, 0)$.

For $n \ge 2$, we continue this process to obtain $\overline{\alpha}_{2n-1}$ (respectively $\overline{\alpha}_{2n}$) by the Intermediate Value Theorem, applied to the function

$$\tilde{\tau}_{2n-1}(\alpha) = \check{\tau}_+(\tau_2,\alpha) - \hat{\tau}_{2n-1}(\tau_0,\alpha)$$

on the whole interval $[0,\overline{\alpha}_{2n-2})$ (respectively $\alpha \mapsto \tilde{\tau}_{2n}(\alpha) = \check{\tau}_-(\tau_2,\alpha) - \hat{\tau}_{2n}(\tau_0,\alpha)$ on the whole interval $[0,\overline{\alpha}_{2n-1})$).

For $\alpha \in (\overline{\alpha}_{2n-1},\overline{\alpha}_{2n-2})$ (respectively $\alpha \in (\overline{\alpha}_{2n},\overline{\alpha}_{2n-1})$), $\mathcal{C}^1_{\tau_0,\alpha}$ remains into the domain delimited by $\mathcal{C}^1_{\tau_0,\overline{\alpha}_{2n-1}} \cup \mathcal{C}^1_{\tau_0,\overline{\alpha}_{2n-2}}$ (respectively $\mathcal{C}^1_{\tau_0,\overline{\alpha}_{2n}} \cup \mathcal{C}^1_{\tau_0,\overline{\alpha}_{2n-1}}$) and necessarily converges to $(\tau_3,0)$ (respectively $(\tau_1,0)$).

Let us prove now (4.1). Setting $\hat{\tau}_0(\tau_0,\overline{\alpha}_n) = \tau_0$, $\hat{\tau}_{n+1}(\tau_0,\overline{\alpha}_n) = \tau_2$ and defining $\tau \mapsto V^{(k)}_{\overline{\alpha}_n}(\tau)$ the k^{th} piece of the $C^1_{\tau_0,\overline{\alpha}_n}$ in the half plane $V > 0$ or $V < 0$, such function is defined on the interval $[\hat{\tau}_{k-1}(\tau_0,\overline{\alpha}_n),\hat{\tau}_k(\tau_0,\overline{\alpha}_n)]$ (or $[\hat{\tau}_k(\tau_0,\overline{\alpha}_n),\hat{\tau}_{k-1}(\tau_0,\overline{\alpha}_n)]$. Then, the integration of (3.4) and (3.5) over these intervals gives

$$\overline{\alpha}_n\,\lambda \sum_{k=0}^{n-1} \Big| \int_{\hat{\tau}_k(\tau_0,\overline{\alpha}_n)}^{\hat{\tau}_{k+1}(\tau_0,\overline{\alpha}_n)} \beta(\tau)\,|V^{(k+1)}_{\overline{\alpha}_n}(\tau)|^{q+1}\,d\tau \Big| = G(\tau_0,\lambda,\tau_2). \tag{4.10}$$

Now, since

$$\int_{\hat{\tau}_{2\,l}(\tau_0,\overline{\alpha}_n)}^{\hat{\tau}_{2\,l+1}(\tau_0,\overline{\alpha}_n)} \beta(\tau)\,|V^{(2\,l+1)}_{\overline{\alpha}_n}(\tau)|^{q+1}\,d\tau \geq \int_{\underline{\tau}}^{\tau_2} \beta(\tau)\tilde{V}_0(\tau)^{q+1}d\tau = \gamma > 0, \tag{4.11}$$

where $\tilde{V}_0$ here is the function corresponding to $C^2_{\tau_2,0}$ in the half plane $V \geq 0$. Then, injecting (4.11) in (4.10) we obtain

$$\frac{1}{2}\,n\,\gamma\,\lambda\,\overline{\alpha}_n(\tau_0,\lambda) \leq G(\tau_0,\lambda,\tau_2),$$

which implies (4.1).

5. Kinetic relation

One important consequence of the existence of several nonclassical traveling waves connecting τ_0 to τ_2 is the following one : a kinetic function (if it exists) should be multivalued, thus would lead to an ill-posed Riemann problem admitting several admissible solutions. It is known from Bedjaoui and LeFloch [4] that, for $q = 0$ and for 2-shock waves at least, such a phenomenon does not arise in the case of a single inflection point. Indeed we were able to define a unique kinetic relation. This result was later extended in Bedjaoui and LeFloch [7] to all $q \geq 0$, in the context of scalar conservation laws whose flux-function admits one inflection point. (This scalar setting is very similar to the system (1.1) for $\lambda > 0$.)

However, the kinetic function associated with 2-shock waves, obtained in [4] for $q = 0$ is *not* monotone, at least near the so-called Maxwell line and for all $\alpha > 0$. This implies that its inverse function (which is nothing but the kinetic function for 1-shock waves) is a *multivalued* function. Recall also that Benzoni [8] noticed, also for van der Waals fluids, that this kinetic function takes two values for $0 < \alpha << 1$.

In contrast with the case of a single inflexion flux the novelty of the present paper is that, for van der Waals-type pressure (with two inflexion points), this phenomenon of multivalued kinetic function not only exists for the 1-shock wave ($\lambda < 0$), but also arises with the 2-shock wave ($\lambda > 0$), and for arbitrarily small values of α. We also cover general exponents $q \geq 0$. More precisely, to each state τ_0 close to the Maxwell line we can associate many phase transitions, for $\lambda > 0$, issuing from τ_0 and arriving at different points τ_2, for $\alpha << 1$, as we will see below.

Introduce the two Maxwell states $m < M$ characterized by

$$p(m) = p(M) \quad \text{and} \quad G(m,0,M) = 0. \tag{5.1}$$

Fix $\tau_0 \in [m-\epsilon, m)$ such that there exist $\lambda_{max} = \lambda_{max}(\tau_0)$ as well as a (first) point $\tau_2(\lambda_{max}) > \tau_0$ such that

$$G(\tau_0,\lambda_{max},\tau_2(\lambda_{max})) = G_\tau(\tau_0,\lambda_{max},\tau_2(\lambda_{max})) = 0. \tag{5.2}$$

Then, it is clear from the graph of the pressure function that, for all $\lambda \in I_{\tau_0} := (0,\lambda_{max})$, there is exactly four intersection points of the line $\tau \mapsto p(\tau_0) - \lambda^2\,(\tau-\tau_0)$ with the graph of $\tau \mapsto p(\tau)$.

These points depend on λ (and, of course, on τ_0) and can be ordered as follows:

$$\tau_0 < \tau_1(\lambda) < \tau_2(\lambda) < \tau_3(\lambda).$$

Now, given an integer $N \geq 1$ let us fix λ_N in the interval I_{τ_0}. For example $\lambda_N := \lambda_{max}/2$, and then fix once for all the ratio α in the system (1.1) to be

$$\alpha := \overline{\alpha}_N(\tau_0, \lambda_N). \tag{5.3}$$

For each integer $k \geq 0$, the function $\lambda \mapsto \overline{\alpha}_k(\tau_0, \lambda)$ is continuous on the interval I_{τ_0}. In addition, from (5.2) it follows that $\overline{\alpha}_0(\tau_0, \lambda_{max}) = 0$, and since

$$\lim_{\lambda \to \lambda_{max}} \overline{\alpha}_0(\tau_0, \lambda) = \overline{\alpha}_0(\tau_0, \lambda_{max}) = 0$$

it follows also that

$$\lim_{\lambda \to \lambda_{max}} \overline{\alpha}_k(\tau_0, \lambda) = 0.$$

Now, in view of the inequalities $0 < \alpha = \overline{\alpha}_N(\tau_0, \lambda_N) < \overline{\alpha}_{N-1}(\tau_0, \lambda_N)$, by the intermediate value theorem there must exists some value λ_{N-1},

$$\lambda_N < \lambda_{N-1} < \lambda_{max}$$

such that $\alpha = \overline{\alpha}_{N-1}(\tau_0, \lambda_{N-1})$. Continuing this construction inductively we can find $N+1$ values,

$$0 < \lambda_N < \lambda_{N-1} < ... < \lambda_0 < \lambda_{max},$$

such that for all $0 \leq k \leq N$, there exists a traveling wave connecting τ_0 to $\tau_2(\lambda_k)$ and correspondong to a trajectory with k oscillations in the (y, τ)-plane.

The same result remains valid if one replaces (6.3) by any α such that

$$0 < \alpha \leq \overline{\alpha}_N(\tau_0, \lambda_N).$$

In conclusion, the non-uniqueness exhibited here occurs near the Maxwell line, for arbitrarily large values of N and when α is sufficiently small.

6. Explicit formulas for the case $q = 1$

In this section, we consider the particular case of a Von Neumann diffusion for which $\beta(\tau)$ always equals 1 and $q = 1$ in system (1.1). The first-order system (1.9) satisfied by the traveling wave solutions now reads

$$\begin{aligned} \tau_y &= v, \\ v_y &= -\lambda\, \alpha\, |v| v + G_\tau(\tau_0, \lambda, \tau). \end{aligned} \tag{6.1}$$

Again and as long as v keeps a constant sign, equation (6.1) is equivalently rewritten as

$$v(\tau)\, \frac{dv}{d\tau}(\tau) + \alpha\, \lambda\, |v(\tau)| v(\tau) = G_\tau(\tau_0, \lambda, \tau), \tag{6.2}$$

that is, if $v \geq 0$:

$$\frac{d}{d\tau}(\frac{v(\tau)^2}{2}) + 2\, \alpha\, \lambda\, \frac{v(\tau)^2}{2} = G_\tau(\tau_0, \lambda, \tau). \tag{6.3}$$

Now multiplying this equation by $e^{2\lambda\alpha\tau}$ and using boundary condition $v(\tau_0) = 0$, one easily obtains the following explicit formula :

$$v(\tau) = e^{-\lambda\alpha\tau} \sqrt{2 \int_{\tau_0}^{\tau} e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s) ds}. \tag{6.4}$$

When $v \leq 0$, we get (in a similar way) :

$$v(\tau) = -e^{\lambda\alpha\tau} \sqrt{2 \int_{\tau_0}^{\tau} e^{-2\lambda\alpha s} G_\tau(\tau_0, \lambda, s) ds}. \tag{6.5}$$

Given τ_0 and λ, let us now precise $\overline{\alpha}_0 = \overline{\alpha}_0(\tau_0, \lambda) > 0$ introduced in Theorem 3.3. By definition, $\overline{\alpha}_0$ is the unique value of α for which there is a (unique) monotone increasing traveling wave connecting τ_0 to τ_2. Thus, the corresponding trajectory satisfies (6.4) with $\alpha = \overline{\alpha}_0$, and in addition condition $v(\tau_2) = 0$ writes

$$\int_{\tau_0}^{\tau_2} e^{2\lambda\overline{\alpha}_0 s} G_\tau(\tau_0, \lambda, s)ds = 0. \tag{6.6}$$

The next statement shows that (6.6) actually selects $\overline{\alpha}_0$.

Lemma 6.1. *The value $\overline{\alpha}_0$ for which there is a monotone connection between τ_0 and τ_2 is the unique zero of function g defined by*

$$\begin{aligned} g : [0, +\infty[\quad &\to \mathbb{R} \\ \alpha \quad &\to g(\alpha) = \int_{\tau_0}^{\tau_2} e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds. \end{aligned} \tag{6.7}$$

Proof. By (2.4), we first observe that

$$g(0) = \int_{\tau_0}^{\tau_2} G_\tau(\tau_0, \lambda, s)ds = G(\tau_0, \lambda, \tau_2) \geq 0.$$

Let be given $\varepsilon > 0$ small enough so that $\tau_1 + \varepsilon < \tau_2$. Then, thanks to (2.2) we have

$$\begin{aligned} g(\alpha) &= \int_{\tau_0}^{\tau_1} e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds + \int_{\tau_1}^{\tau_2} e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds \\ &\leq e^{2\lambda\alpha\tau_1} \int_{\tau_0}^{\tau_1} G_\tau(\tau_0, \lambda, s)ds + \int_{\tau_1+\varepsilon}^{\tau_2} e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds \\ &\leq e^{2\lambda\alpha\tau_1}\Big(\int_{\tau_0}^{\tau_1} G_\tau(\tau_0, \lambda, s)ds + e^{2\lambda\alpha\varepsilon} \int_{\tau_1+\varepsilon}^{\tau_2} G_\tau(\tau_0, \lambda, s)ds\Big). \end{aligned} \tag{6.8}$$

From (2.2) and (6.8), it is now clear that $\lim_{\alpha\to+\infty} g(\alpha) = -\infty$ so that equation $g(\alpha) = 0$ admits at least one solution. By differentiation of g, we get

$$\begin{aligned} g'(\alpha) &= 2\lambda \int_{\tau_0}^{\tau_2} s\, e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds \\ &= 2\lambda\Big(\int_{\tau_0}^{\tau_1} s\, e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s)ds + \int_{\tau_1}^{\tau_2} s\, e^{2\lambda\alpha s} G_\tau(\tau_0, \lambda, s\Big)ds) \\ &< 2\lambda\,\tau_1\, g(\alpha), \end{aligned}$$

that is to say $g'(\alpha) < 0$ as soon as $g(\alpha) \leq 0$. Solution of $g(\alpha) = 0$ is then necessarily unique. □

7. Numerical experiments

Our objective is now to illustrate numerically existence of trajectories with multiple changes of monotonicity (as stated in Theorem 4.1) when using explicit formulas of previous section. We consider, for simplification, a quadri-linear pressure law :

$$p(\tau) = \begin{cases} -7\tau + 10 & \text{if} \quad \tau \leq 1, \\ 4\tau - 1 & \text{if} \quad 1 < \tau \leq 2, \\ -\frac{5}{2}\tau + 12 & \text{if} \quad 2 < \tau \leq 4, \\ -\frac{1}{5}\tau + \frac{14}{5} & \text{if} \quad \tau > 4, \end{cases} \tag{7.1}$$

which is, to some extent, the simplest case of van der Waals type function. As a left-hand state, we fix $\tau_0 = 0.5$ and study trajectories issuing from τ_0, propagating with a positive speed λ, and arriving at the equilibrium point τ_2. To ensure illustration, we represent on Figure 1 the graph of pressure p together with the line $\tau \to d(\tau) = p(\tau_0) - \lambda^2(\tau - \tau_0)$ for several values of λ, namely $0.80, 0.85$ and 0.90. Observe that these values actually yield to four intersection points with pressure p, while compatibility condition (2.4), with strict inequality, holds true. More precisely, when $\lambda = 0.85$ we get

$$\tau_0 = 0.5, \quad \tau_1 \simeq 1.6646, \quad \tau_2 \simeq 2.8910, \quad \tau_3 \simeq 7.7727,$$

and

$$\int_{\tau_0}^{\tau_2} (p(\tau_0) - p(s) - \lambda^2(s - \tau_0))\, ds \simeq 0.8566 > 0.$$

From now on, λ is set to be 0.85.

FIGURE 1. Pressure p and line $\tau \to d(\tau)$ for several values of λ

7.1. **Monotone traveling wave (0 oscillation).** In the phase plane, we are looking for $(\tau, v(\tau))$ the unique monotone increasing trajectory (in (y, τ)) connecting τ_0 to τ_2, for which we have an explicit formula given by (6.4) with $\alpha = \overline{\alpha}_0$, that is

$$v(\tau) = e^{-\lambda\overline{\alpha}_0\tau}\sqrt{2\int_{\tau_0}^{\tau} e^{2\lambda\overline{\alpha}_0 s} G_\tau(\tau_0, \lambda, s) ds} \quad \text{for} \quad \tau_0 \le \tau \le \tau_2.$$

Now we have to determine $\overline{\alpha}_0$, which is done by numerically solving $g(\alpha) = 0$ (see (6.7) and Lemma 6.1). We have used a Newton-Raphson algorithm to obtain $\overline{\alpha}_0 \simeq 0.32821$. Figure 2 shows both the graph of function g for $\alpha \in [0, 1]$ (Left) and the monotone traveling wave connecting τ_0 to τ_2 in the phase plane (Right).

FIGURE 2. Function g - Trajectory without oscillation ($\lambda = 0.85$)

7.2. **Traveling wave with one oscillation.**
Let us now address the case of a 1-oscillation trajectory connecting τ_0 and τ_2. As stated in Theorem 3.4, it is obtained when taking $\alpha = \overline{\alpha}_1$. Recall that $\overline{\alpha}_1$ is necessary less than $\overline{\alpha}_0$. In order to determine $\overline{\alpha}_1$, we propose to numerically solve equation $\hat{\tau}_1(\tau_0, \alpha) = \check{\tau}_+(\tau_2, \alpha)$ as it is suggested

in proof of Theorem 3.4. Such values are characterized by $v(\hat{\tau}_1(\tau_0,\alpha)) = v(\check{\tau}_+(\tau_2,\alpha)) = 0$, where

$$v(\tau) = \begin{cases} e^{-\lambda\alpha\tau}\sqrt{2\int_{\tau_0}^{\tau} e^{2\lambda\alpha s} G_\tau(\tau_0,\lambda,s)ds} & \text{for} \quad \tau_0 \le \tau \le \hat{\tau}_1(\tau_0,\alpha), \ \text{(increasing part)}, \\ -e^{\lambda\alpha\tau}\sqrt{2\int_{\tau_2}^{\tau} e^{-2\lambda\alpha s} G_\tau(\tau_0,\lambda,s)ds} & \text{for} \quad \tau_2 \le \tau \le \check{\tau}_+(\tau_2,\alpha), \ \text{(decreasing part)}. \end{cases}$$

The corresponding trajectory in the plane (y,τ), for $\alpha = \overline{\alpha}_1$ is monotone increasing ($v \ge 0$) from τ_0 to $\hat{\tau}_1(\tau_0,\overline{\alpha}_1)$ and then monotone decreasing ($v \le 0$) from $\check{\tau}_+(\tau_2,\overline{\alpha}_1) = \hat{\tau}_1(\tau_0,\overline{\alpha}_1)$ to τ_2. By means of a Newton-Raphson method, we found $\overline{\alpha}_1 \simeq 0.00773$ and $\hat{\tau}_1(\tau_0,\overline{\alpha}_1) = \check{\tau}_+(\tau_2,\overline{\alpha}_1) \simeq 12.2326$. Note that the increasing part is nothing but (6.4) with $\alpha = \overline{\alpha}_1$, while the decreasing part coincides with (6.5) with $\alpha = \overline{\alpha}_1$ but with boundary condition $v(\tau_2) = 0$, i.e. we replace τ_0 by τ_2. On Figure 3, functions $\alpha \to \hat{\tau}_1(\tau_0,\alpha)$ and $\alpha \to \check{\tau}_+(\tau_2,\alpha)$ are first displayed (Left), and then the traveling wave connecting τ_0 to τ_2 with one oscillation in the phase plane (Right). Observe that functions $\hat{\tau}_1$ and $\check{\tau}_+$ actually intersect once and satisfy monotonicity properties stated in Proposition 3.5.

FIGURE 3. Functions $\hat{\tau}_1$ and $\check{\tau}_+$ - Trajectory with one oscillation ($\lambda = 0.85$)

7.3. **Traveling wave with two oscillations.**

Finally, we illustrate the case of a 2-oscillation trajectory from τ_0 to τ_2 obtained with $\alpha = \overline{\alpha}_2 < \overline{\alpha}_1$. As seen in proof of Theorem 4.1, $\overline{\alpha}_2$ is solution of $\hat{\tau}_2(\tau_0,\alpha) = \check{\tau}_-(\tau_2,\alpha)$ where $\hat{\tau}_2$ (respectively $\check{\tau}_-$) is now the second intersection point of trajectory coming from τ_0 and lying initially in the quadrant $Q^1_{\tau_0}$ (respectively the last intersection point of trajectory arriving at τ_2 in the quadrant $Q^2_{\tau_2}$ quadrant) with the τ-axis. Such values are characterized by $v(\hat{\tau}_1(\tau_0,\alpha)) = v(\hat{\tau}_2(\tau_0,\alpha)) = v(\check{\tau}_-(\tau_2,\alpha)) = 0$, where function v is now defined by

$$v(\tau) = \begin{cases} e^{-\lambda\alpha\tau}\sqrt{2\int_{\tau_0}^{\tau} e^{2\lambda\alpha s} G_\tau(\tau_0,\lambda,s)ds} & \text{for} \quad \tau_0 \le \tau \le \hat{\tau}_1(\tau_0,\alpha), \ \text{(first increasing part)}, \\ -e^{\lambda\alpha\tau}\sqrt{2\int_{\hat{\tau}_1}^{\tau} e^{-2\lambda\alpha s} G_\tau(\tau_0,\lambda,s)ds} & \text{for} \quad \hat{\tau}_2(\tau_0,\alpha) \le \tau \le \hat{\tau}_1(\tau_0,\alpha), \ \text{(decreasing part)}, \\ e^{-\lambda\alpha\tau}\sqrt{2\int_{\tau_2}^{\tau} e^{2\lambda\alpha s} G_\tau(\tau_0,\lambda,s)ds} & \text{for} \quad \check{\tau}_-(\tau_2,\alpha) \le \tau \le \tau_2, \ \text{(second increasing part)}. \end{cases}$$

Numerically, we found $\overline{\alpha}_2 \simeq 0.00748$ and $\hat{\tau}_2(\tau_0,\overline{\alpha}_2) = \check{\tau}_-(\tau_2,\overline{\alpha}_2) \simeq 1.01944$. In addition, the first intersection point of the trajectory with line $v = 0$ is now $\hat{\tau}_1(\tau_0,\overline{\alpha}_2) \simeq 12.2387$ and we note that $\hat{\tau}_1(\tau_0,\overline{\alpha}_2) > \hat{\tau}_1(\tau_0,\overline{\alpha}_1)$ according to decreasing monotonicity property of function $\alpha \to \hat{\tau}_1(\tau_0,\alpha)$ established in Proposition 3.5. Then, the trajectory being monotone increasing ($v \ge 0$) from τ_0 to $\hat{\tau}_1(\tau_0,\overline{\alpha}_2)$, then monotone decreasing ($v \le 0$) from $\hat{\tau}_1(\tau_0,\overline{\alpha}_2)$ to $\hat{\tau}_2(\tau_0,\overline{\alpha}_2)$, and finally monotone increasing ($v \ge 0$) from $\check{\tau}_-(\tau_2,\overline{\alpha}_2)$ to τ_2. On Figure 4, we plotted functions $\alpha \to \hat{\tau}_2(\tau_0,\alpha)$ and $\alpha \to \check{\tau}_-(\tau_2,\alpha)$ on the interval $[0,\overline{\alpha}_1[$ (Left), and the traveling wave connecting τ_0 to τ_2 with two oscillations in the phase plane (Right). Again, we note that functions $\hat{\tau}_2$ and $\check{\tau}_-$ actually intersect once (near $\overline{\alpha}_1$) and obey monotonicity properties stated in Proposition 4.2.

FIGURE 4. Functions $\hat{\tau}_2$ and $\check{\tau}_-$ - Trajectory with two oscillations ($\lambda = 0.85$)

To conclude, we propose to leave the phase plane (τ, v) in order to observe the behavior of the three trajectories considered in this section in the plane (y, τ). The traveling wave without oscillation (respectively with one oscillation) is represented on Figure 5 - Left (respectively Figure 5 - Right). The traveling wave with two oscillations is represented on Figure 6 - Left while Figure 6 - Right stacks the three trajectories.

FIGURE 5. Trajectories in the plane (y, τ) (First part)

Figure 6. Trajectories in the plane (y, τ) (Second part)

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