Chapitre 1

The Riemann problem for the multi-pressure Euler system

Ce chapitre fait l'objet d'un article en préparation.

The Riemann problem for the multi-pressure Euler system

Chalons C.[1, 2] and Coquel F.[3]

Résumé

We prove existence and uniqueness of Riemann solutions for the Euler equations closed by N independent constitutive pressure laws. This PDE model stands as a natural asymptotic system for the multi-pressure Navier-Stokes equations in the regime of an infinite Reynolds number. Due to the inherent lack of conservation form in the viscous regularization, the limit system exhibits measure valued source terms concentrated on shock discontinuities. These non positive bounded measures, called kinetic relations after [15], are known to provide a suitable tool to encode the small scale sensitiveness in the singular limit. Here considering N independent polytropic pressure laws, we show that these kinetic relations can be derived when solving a simple algebraic problem which governs the endpoints of the underlying viscous shock profiles for any given but prescribed ratios of viscosities in the viscous perturbations. The analysis based on travelling wave solutions will allow us to introduce the asymptotic Euler system in the setting of piecewise Lipschitz-continuous functions so as to study the Riemann problem.

1.1 Introduction

The present work aims at studying a particular class of solutions of the following non linear hyperbolic system with viscous perturbations :

$$\left\{\begin{array}{l} \partial_t\rho_\epsilon + \partial_x(\rho u)_\epsilon = 0, \\ \partial_t(\rho u)_\epsilon + \partial_x(\rho u^2 + \sum_{i=1}^N p_i)_\epsilon = \epsilon\partial_x((\sum_{i=1}^N \mu_i)\partial_x u)_\epsilon, \\ \partial_t(\rho\varepsilon_i)_\epsilon + \partial_x(\rho\varepsilon_i u)_\epsilon + (p_i)_\epsilon\partial_x u_\epsilon = \epsilon(\mu_i)_\epsilon(\partial_x u)_\epsilon^2, \quad i = 1, ..., N, \end{array}\right. \tag{1.1}$$

in the asymptotic regime of a vanishing parameter $\epsilon \to 0^+$, *i.e.* in the regime of an infinite Reynolds number. This PDE system indeed stands as a natural extension of the usual Navier-Stokes equations when considering N independent partial internal energies. The N independent pressure laws p_i are given hereafter polytropic gas definitions.

This work extends a study by Berthon [10] in the case $N = 2$ for finite Reynolds numbers (*i.e.* for fixed $\epsilon > 0$), motivated by some turbulent models from the physics (the so-called $k - \epsilon$ or associated models). The present extension to values $N > 2$ is motivated by more sophisticated turbulent modelling where several turbulent temperatures k_i with $i = 1, ..., N$ are distinguished for the sake of a better accuracy. The PDE system (1.1) can also be understood as a simplified form of complex mixture flow modelling : namely advection equations for species mass fractions are in order. Such

1. O.N.E.R.A., B.P. 72, 29 avenue de la Division Leclerc, 92322 Châtillon Cedex, France.
E-mail: Christophe.Chalons@onera.fr.
2. Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, U.M.R. 7641, École Polytechnique, 91128 Palaiseau Cedex, France.
E-mail: chalons@cmapx.polytechnique.fr
3. Laboratoire JLL & Centre National de la Recherche Scientifique, U.M.R. 7598, Boîte courrier 187, 75252 Paris Cedex 05, France.
E-mail: coquel@ann.jussieu.fr

models are discussed for instance by Truesdell [137] or more recently by Lagoutière [79]. All these models are oftenly addressed in practical issues in the regime of large Reynolds numbers (*i.e.* small values of ϵ). Since we consider an infinite domain (*i.e.* without boundary), we are classically led to neglect the small scale effects coming with small values of ϵ in the solutions of (1.1) : that is to say one has to pass in the limit $\epsilon \to 0$. When a single pressure law p_1 is addressed, this procedure formally gives rise to the usual 3×3 Euler equations for governing the conservative unknowns : density ρ, momentum ρu and total energy $\rho E := (\rho u)^2/2\rho + \rho\varepsilon_1$. A full convergence result is still an open question. In the extended framework of N independent pressure laws p_i, the situation turns out to be one step further difficult because of the apparent lack of conservation form. As underlined in Chalons [19] (see also Berthon [10] and below for a brief discussion), the PDE system (1.1) cannot recast in full conservation form without restrictive modelling assumptions. Let us briefly put forward the consequences of such a negative issue.

As already claimed, the first order underlying system obtained when formally setting $\epsilon = 0$ in (1.1) is seen hereafter to be hyperbolic with two extreme genuinely non linear fields. By this property, we thus expect families of smooth solutions $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ of (1.1) to converge generally speaking to discontinuous functions as ϵ goes to zero. Because of the lack of a conservation form for (1.1), the difficulty is to understand these limit functions as solutions of a first order limit system. Our main objective here is to derive such a limit system when restricting ourselves to piecewise Lipschitz-continuous limit functions. A notion of weak solutions for such functions will naturally arise following the pionnering works by LeFloch [90]. The main result of this paper is the existence and uniqueness of a particular class of such weak solutions of practical importance : namely solutions of the Riemann problem coming with the limit system. Non linear hyperbolic systems in non conservation form have received some important contributions over the past decade. LeFloch, Dal Maso and Murat [42] (see also LeFloch [90]) have in particular proposed a notion of fixed family of paths to define ambiguous products, extending the work by Volpert [145]. Equipped with this definition, they have been able to propose a notion of weak solutions for such systems and have solved the Riemann problem for general systems when considering two sufficiently close initial states. Of course and for two fixed initial states, the solution under consideration heavily depends on the particular family of paths under consideration. LeFloch has proved in [90] that the choice of a particular family of paths can be performed on the ground of an analysis of the travelling wave solutions of a parabolic regularization of the hyperbolic system under consideration. Because of the non conservative nature of the first order system, the derived family of paths generally varies when considering non proportionnal viscous tensors. This sensitiveness has been exemplified for instance by Raviart and Sainsaulieu [112] within the frame of distinct averaged multi-phase flows models and also by Berthon [10] for various closures of turbulent models. Such a sensitiveness has been proved to also occur in completely different settings : mixed hyperbolic-elliptic systems (see Bedjaoui and LeFloch [9], Chalons and LeFloch [25], ...) or even hyperbolic systems in pure conservation form but with a loss of non linearity (see LeFloch [90], [87], Hayes and LeFloch [69], [65] Chalons and LeFloch [24], [26] ...). Here, we adopt this natural strategy to define shock solutions of the limit first order system as limits of travelling wave solutions of (1.1) when $\epsilon \to 0$. Using the Lasalle invariance principle, Berthon and Coquel [14] have proved the existence in the large of travelling wave solutions for (1.1). Their analysis proves by abstract geometrical considerations the existence of an unique exit state $\mathbf{u}_+$ reached in the future by a travelling wave solution connecting some given state $\mathbf{u}_-$ in the past, a suitable speed of propagation σ being prescribed. The existence of the triple $(\sigma; \mathbf{u}_-, \mathbf{u}_+)$ then naturally serves to define a shock solution (see [90], [112], [10] and below). But due to the non conservative nature of the dynamical system governing the heteroclinic solution under consideration, barely little is known about the exit state $\mathbf{u}_+$ except its existence and the property that its precise form heavily depends on the N ratios of the viscosities $\mu_i / \sum_{j=1}^{N} \mu_j$. Such a dependence simply reflects the sensitiveness we have already reported. In other words, shock solutions are known to exist as regularization depending limit functions, but without precise informations about this dependency. Of course it is out of reach to tackle the study of the Riemann problem without a minimum of informations about shock solutions.

In the present work, we show how to get sufficient informations from the analysis of an auxiliary companion ODE problem. Besides its dimensionless form, the auxiliary ODE system turns out to be linear with constant coefficients: the N viscosity ratios dictate these coefficients, the thermodynamic closures being tacitly fixed. Its solution, therefore depending on these N ratios, will then serve to solve a non linear algebraic scalar problem that characterizes the critical points of the original dynamical system associated with (1.1). From the proposed analysis and roughly speaking, a left state $\mathbf{u}_-$ being prescribed, the state $\mathbf{u}_+$ will come out as a natural function of the speed of propagation σ when just solving a scalar equation for different values of σ (the N viscosity ratios being fixed). On the one hand, the proposed analysis will allow to derive a convenient form of the limit system based on N so-called kinetic functions, after LeFloch [88] (see also [15] for the setting of hyperbolic systems in non conservation form). Roughly speaking, these N kinetic functions stand for the singular limits as ϵ goes to zero of the N entropy dissipation rates entering the N entropy equations associated with (1.1). On the second hand, we will be able to define a notion of shock curves in the physical phase space, which extends naturally the setting of a single pressure law. Rarefaction curves will be characterized without difficulty since their derivation relies on particular smooth solutions for the limit system. We shall be in a position to determine the projection of these two families of curves on the convenient plane with total pressure $(\sum_{i=1}^{N} p_i)$ and velocity (u) as independent coordinates. The strict monotonicity property of these projections and their asymptotic behaviours will allow us to prove existence and uniqueness of a solution to the Riemann problem under the only classical requirement of staying away from vacuum. Of course Riemann solutions will be sensitive with respect to the N viscosity ratios since they are generally made of shock solutions. This small scale sensitiveness will be seen to be encoded in the limit system in the definition of the N kinetic functions.

1.2 Some basic properties of the PDE model

In this section, we precise the required closure assumptions and state useful properties of the solutions of the system (1.1) when choosing $\epsilon = 1$ for simplicity in the notations :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i) = \partial_x ((\sum_{i=1}^N \mu_i) \partial_x u), \\ \partial_t \rho \varepsilon_i + \partial_x \rho \varepsilon_i u + p_i \partial_x u = \mu_i (\partial_x u)^2, \quad i = 1, ..., N. \end{cases} \tag{1.2}$$

The N independent partial pressure laws p_i, $i = 1, ..., N$, entering (1.2) find the following polytropic ideal gas definitions :

$$p_i = (\gamma_i - 1)\rho\varepsilon_i, \quad i = 1, ..., N,$$

where the N adiabatic exponents γ_i are constant real numbers with $\gamma_i > 1$. Then, the viscosity coefficients μ_i are assumed to be non negative real numbers but with the requirement that

$$\mu := \sum_{i=1}^N \mu_i > 0.$$

The unknown $\mathbf{u} = (\rho, \rho u, \{\rho\varepsilon_i\}_{i=1,...,N})$ of (1.2) is associated with the following natural phase space :

$$\Omega := \{\mathbf{u} = (\rho, \rho u, \{\rho\varepsilon_i\}_{i=1,...,N}) \in I\!R^{N+2},\ \rho > 0,\ \rho u \in I\!R,\ \rho\varepsilon_i > 0,\ i = 1, ..., N\}, \tag{1.3}$$

from which the vacuum is excluded for simplicity. We first state the following easy result which proof is left to the reader (see Berthon [10]) :

Proposition 1 *The first order underlying system in (1.2) :*

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i) = 0, \\ \partial_t \rho \varepsilon_i + \partial_x \rho \varepsilon_i u + p_i \partial_x u = 0, \quad i = 1, ..., N, \end{cases} \tag{1.4}$$

is hyperbolic over Ω and admits the following three distinct eigenvalues :

$$u - c, \quad u, \quad u + c, \quad c^2 = \sum_{i=1}^N \frac{\gamma_i p_i}{\rho},$$

where u has N order of multiplicity. Moreover, the eigenvalues $u - c$ and $u + c$ are associated with genuinely non linear fields, while u is associated with linearly degenerate fields. In addition, the latter admits the velocity u and the total pressure $p := \sum_{i=1}^N p_i$ as Riemann invariants.

By the non linearity of the two extreme fields, solutions of the hyperbolic system (1.4) are known to generally develop discontinuities in finite time. The system (1.4) must be thus understood in a weaker sense, but the non conservative products in (1.4) are (in general) ambiguous in the usual sense of the distributions. Notice that the intermediate fields are naturally associated with discontinuous solutions, namely the so-called contact discontinuities propagating with constant speed u by the linear degeneracy of these fields (see also subsection 1.4.2 for related details). The continuity of the velocity u thus precludes the occurence of ambiguities in all the non conservative products entering (1.4) (but also in (1.2)). The situation is completely different when addressing discontinuities coming with the two extreme fields : namely the so-called shock discontinuities. Indeed and across shocks, the velocity u and the partial pressures p_i will be seen below to be necessarily discontinuous. Therefore, passing into the limit $\epsilon \to 0$ in (1.1) will necessarily result into non trivial Borel measures in both the left and right hand sides of the N last equations. This last observation thus motivates us to study for existence additional conservation laws for smooth solutions of (1.2). The next statement gives the existence of solely one additional non trivial conservation law. Indeed, the N supplementary balance equations exhibited below dot not give rise to conservation laws unless specific assumptions on the adiabatic exponents and/or on the viscosity coefficients are made (see Berthon [10] and Chalons [19] for the details).

Proposition 2 *Smooth solutions of (1.2) obey the following conservation law on the total energy ρE :*

$$\partial_t \rho E + \partial_x (\rho E + p) u = \partial_x (\mu u \partial_x u), \tag{1.5}$$

where we have set :

$$\rho E = \frac{(\rho u)^2}{2\rho} + \rho\varepsilon, \quad \rho\varepsilon = \sum_{i=1}^{N} \rho\varepsilon_i, \tag{1.6}$$

and

$$p = \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} (\gamma_i - 1)\rho\varepsilon_i.$$

Moreover, smooth solutions of (1.2) obey the following entropy balance equations on the specific entropies s_i :

$$\partial_t \rho s_i + \partial_x \rho s_i u = \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i - 1}} (\partial_x u)^2, \quad i = 1, ..., N, \tag{1.7}$$

where we have set :

$$s_i = \frac{p_i}{\rho^{\gamma_i}}, \quad i = 1, ..., N. \tag{1.8}$$

The $(N+1)$ laws (1.5) and (1.7) are the only additional non trivial equations, up to some classical non linear transforms in the $\{s_i\}_{i=1,...,N}$.

The $(N+2)$ PDE system (1.2) cannot be thus given an admissible change of variables so as to recast in full conservation form. The reported properties of the shock solutions clearly make the equations (1.7), once rescaled in time and space by the parameter ϵ, to give birth to non trivial measures in the limit $\epsilon \to 0$ as soon as $\mu_i > 0$. In other words and whatever are the variables under consideration, passing to the limit in the rescaled system (1.1) will result in the need for dealing with measures concentrated on shock discontinuities. According to the motivations discussed in the introduction, we propose to define the shock solutions when passing into the limit of travelling wave solutions of system (1.1) as ϵ goes to zero. This procedure will allow for defining the measures entering the limit first order system coming from (1.1) as $\epsilon \to 0$. As it is well-known, a travelling wave solution of system (1.2) immediately gives, once rescaled by ϵ, a corresponding travelling wave solution of (1.1) (see hereafter). As an immediate consequence, it thus suffices to study the travelling wave solutions of (1.2).

1.3 Travelling wave solutions

Let us now consider a travelling wave solution of (1.2), that is to say a smooth solution of (1.2), we denote $\mathbf{u}$, under the special form :

$$\left\{ \mathbf{u}(x,t) = \mathbf{u}(\xi), \quad \xi = x - \sigma t, \right. \tag{1.9}$$

where $\sigma \in I\!R$ stands for the speed of propagation. Such a solution has to obey the following usual asymptotic behaviour :

$$\lim_{\xi \to -\infty} \mathbf{u}(\xi) = \mathbf{u}_-, \qquad \lim_{\xi \to +\infty} \mathbf{u}(\xi) = \mathbf{u}_+, \tag{1.10}$$

for two constant states $\mathbf{u}_-$ and $\mathbf{u}_+$ in Ω. When considering the system (1.2), the solution $\mathbf{u}(\xi)$ under consideration must satisfy the following non linear $(N+2)$-ODE system :

$$\left\{ \begin{array}{l} -\sigma d_\xi \rho + d_\xi \rho u = 0, \\ -\sigma d_\xi \rho u + d_\xi(\rho u^2 + p) = d_\xi(\mu d_\xi u), \\ -\sigma d_\xi \rho \varepsilon_i + d_\xi \rho \varepsilon_i u + p_i d_\xi u = \mu_i (d_\xi u)^2, \quad i = 1, ..., N, \end{array} \right. \tag{1.11}$$

where d_ξ represents the classical derivative operator. Observe from the first ODE in (1.11) that the relative mass flux $m = \rho(\xi)(u(\xi) - \sigma)$ is an algebraic invariant given by $m = \rho_-(u_- - \sigma)$. In the classical setting of a single pressure law, this algebraic invariant cannot be zero (see below) and is positive (respectively negative) for viscous shock profiles associated with the first field (respectively the third field). Classical considerations within this setting naturally extend to present one to prove that it suffices to restrict ourselves to the study of viscous profiles for shocks coming with the first field. Indeed, viscous shock profiles for the third field will be recovered when exchanging the role of the exit states $\mathbf{u}_-$ and $\mathbf{u}_+$ while reversing the sign of ξ in the forthcoming analysis. On the ground of this observation, let us now introduce a convenient reformulation of the dynamical system (1.11) when considering positive values of m :

Lemma 1 *Let be given a fixed state $\mathbf{u}_-$ in Ω and a speed σ with $m = \rho_-(u_- - \sigma) > 0$. Then, travelling wave solutions of (1.11) equivalently obey the following $(N+1)$-ODE system :*

$$\left\{ \begin{array}{l} d_\xi \tau = \frac{1}{\mu m} \mathcal{F}(\tau, \{s_i\}_{i=1,...,N}), \\ d_\xi s_i = (\gamma_i - 1)\tau^{\gamma_i - 1} \frac{\mu_i}{\mu^2 m} \mathcal{F}^2(\tau, \{s_i\}_{i=1,...,N}), \quad i = 1, ..., N, \end{array} \right. \tag{1.12}$$

where we have set :

$$\mathcal{F}(\tau, \{s_i\}_{i=1,...,N}) = m^2(\tau - \tau_-) + (p(\tau, \{s_i\}_{i=1,...,N}) - p_-). \tag{1.13}$$

Consequently, critical points $\mathbf{u}_+$ of the dynamical system (1.12) are entirely characterized by :

$$\mathcal{F}(\tau_+, \{(s_i)_+\}_{i=1,...,N}) = 0. \tag{1.14}$$

Before addressing the proof of this lemma, notice that (1.14) just reflects the conservation form of the PDE governing the momentum ρu. Since

$$p(\tau, \{s_i\}_{i=1,...,N}) = \sum_{i=1}^{N} s_i \tau^{-\gamma_i},$$

with $\gamma_i > 1$, the vector field entering the dynamical system (1.12) is Lipschitz-continuous over the following open subset:

$$\mathcal{D} = \{\mathbf{v} := (\tau, \{s_i\}_{i=1,...,N}) \in I\!R^{N+1},\ \tau > 0\}. \tag{1.15}$$

By the well-known continuation theorem, we thus have existence and uniqueness of a trajectory $\mathbf{v}(\xi)$ with $\mathbf{v}(0) = \mathbf{v}_0$ in $\mathcal{D}$ over the maximal interval $[0, \xi_{max}^{+}(\mathbf{v}_0))$, the associated maximal time ξ_{max}^{+} being finite if and only if $\tau(\xi_{max}^{+}) = 0$.

Proof Since by assumption a travelling wave is a smooth solution of (1.2), it must also satisfy the N independent balance equations (1.7) for governing the specific entropies s_i. Consequently, a travelling wave solution equivalently solves:

$$\begin{cases} -\sigma d_\xi \rho + d_\xi \rho u = 0, \\ -\sigma d_\xi \rho u + d_\xi(\rho u^2 + p) = d_\xi(\mu d_\xi u), \\ -\sigma d_\xi \rho s_i + d_\xi \rho s_i u = \mu_i(\gamma_i - 1)\tau^{(\gamma_i - 1)}(d_\xi u)^2, \quad i = 1, ..., N. \end{cases}$$

Next, the algebraic invariant $m = \rho(\xi)(u(\xi) - \sigma)$ gives $m d_\xi \tau = d_\xi u$ so that we get after easy calculations:

$$\begin{cases} m^2 d_\xi \tau + d_\xi p = m d_\xi(\mu d_\xi \tau), \\ m d_\xi s_i = m^2 \mu_i(\gamma_i - 1)\tau^{(\gamma_i - 1)}(d_\xi \tau)^2, \quad i = 1, ..., N. \end{cases} \tag{1.16}$$

Integrating the first equation of (1.16) from $-\infty$ to ξ, we arrive at:

$$\mu m d_\xi \tau = \mathcal{F}(\tau, \{s_i\}_{i=1,...,N}) = m^2(\tau - \tau_-) + (p(\tau, \{s_i\}_{i=1,...,N}) - p_-),$$

so that (1.16) easily reexpresses as (1.12). This completes the proof.

The main result of this section is:

Theorem 1 *Let be given a fixed state $\mathbf{u}_-$ in Ω and let us consider a speed σ subject to the following condition:*

$$\frac{u_- - \sigma}{c_-} > 1, \quad c_-^2 = \sum_{i=1}^{N} \gamma_i (p_i \tau)_-. \tag{1.17}$$

Then (1.17) is a necessary and sufficient condition for existence and uniqueness (up to some translation) of a travelling wave solution.

Remark 1 Observe that this statement extends immediately to viscous profiles for shocks coming with the third field but when asking:

$$\frac{u_+ - \sigma}{c_+} < -1, \quad c_+^2 = \sum_{i=1}^{N} \gamma_i (p_i \tau)_+.$$

In other words and for any given prescribed mass flux $m = \rho_-(u_- - \sigma)$ with $m > (\rho c)_-$, the dynamical system (1.12) admits an unique orbit $\mathcal{O}^-$ connecting $\mathbf{u}_-$ in the past and a (unique) state $\mathbf{u}_+$ in the future. The proof of this claim will follow from the developments proposed in the next three subsections. The first one will establish the existence of exactly one orbit, say $\mathcal{O}^-$, that can connect $\mathbf{u}_-$ in the past. This orbit will be seen to be necessarily compressive in its maximal interval of existence: $d_\xi \tau < 0$ for all finite $\xi < \xi^+_{max}$ with $\xi^+_{max} \in \overline{I\!R}$. This strong but expected property will allow us in the second subsection to derive an auxiliary linear ODE problem with constant coefficients. By construction, its solution will describe all the states $(\tau, \{s_i(\tau)\}_{i=1,...,N})$ along $\mathcal{O}^-$. Equipped with this parametrization, we shall then solve the algebraic equation $\mathcal{F}(\tau, \{s_i(\tau)\}_{i=1,...,N}) = 0$ that governs all the critical points of the dynamical system (1.12). We shall get the existence of an unique relevant $\tau_+ > 0$ so as to define a state $\mathbf{u}_+$ to be connected by $\mathcal{O}^-$ in the future.

1.3.1 Heteroclinic orbits are compressive

Here, the main result is given by the following statement:

Proposition 3 *The state $\mathbf{u}_-$ can be connected in the past by orbits of the dynamical system (1.12) if and only if the Lax condition (1.17) is satisfied. Under this condition, there exists at most one heteroclinic orbit connecting $\mathbf{u}_-$, which in addition is such that the mapping $\xi \to \tau(\xi)$ is necessarily strictly decreasing in the maximal interval of existence.*

In other words, viscous shock profiles are necessarily compressive as soon as they exist. In the usual setting of a single pressure law, the property (1.17) is known to be connected to the familiar Lax shock conditions expressed on the two exit states $\mathbf{u}_-$ and $\mathbf{u}_+$ of a heteroclinic solution. It is worthy to briefly prove for validity this connection in the present extended framework. As seen below, the above statement will follow from the well-known center manifold theorem (see [79] for instance). Indeed, the linearization of the vector field entering the definition of the dynamical system (1.12) under study, when evaluated at the critical point $\mathbf{u}_-$, is easily seen to admit only one non trivial eigenvalue:

$$\lambda(\mathbf{u}_-) = \frac{1}{\mu m}\partial_\tau \mathcal{F}(\mathbf{u}_-) = \frac{(\rho c)^2_-}{\mu m}((\frac{m}{\rho c})^2_- - 1). \tag{1.18}$$

A necessary and sufficient condition for the state $\mathbf{u}_-$ to be connected in the past is then $\lambda(\mathbf{u}_-) > 0$, but this condition is nothing else (1.17). A next and important consequence of the center manifold theorem states that if a critical point $\mathbf{u}_+$ is connected in the future, then necessarily this exit state obeys $\lambda(\mathbf{u}_+) < 0$, that is to say:

$$\frac{u_+ - \sigma}{c_+} < 1, \quad c_+^2 = \sum_{i=1}^{N} \gamma_i (p_i \tau)_+. \tag{1.19}$$

The two inequalities (1.17) and (1.19) thus express nothing but the familiar Lax shock inequalities on the exit states:

$$u_+ - c_+ < \sigma < u_- - c_-. \tag{1.20}$$

We now prove the above result.

Proof An associated eigenvector with the eigenvalue (1.18) is given by e_τ, the unit vector in the τ direction. By the center manifold theorem, the manifold $\mathcal{W}_{\mathbf{u}}(\mathbf{u}_-)$, made of the totality of the orbits connecting $\mathbf{u}_-$ in the past, is necessarily tangent to e_τ at $\mathbf{u}_-$. As a consequence, there exists exactly two orbits connecting $\mathbf{u}_-$ in the past, say $\mathcal{O}^-$ and $\mathcal{O}^+$, with the following distinctive properties, valid for $|\tau - \tau_-|$ sufficiently small :

$$\tau < \tau_- \text{ for } \tau \in \mathcal{O}^-, \quad \text{while} \quad \tau > \tau_- \text{ for } \tau \in \mathcal{O}^+. \tag{1.21}$$

Let us then prove that the orbit $\mathcal{O}^+$ cannot give rise to a heteroclinic solution. In that aim, it suffices to prove that the following domain is positively invariant for the dynamical system (1.12) :

$$\mathcal{D} = \{\mathbf{u} \in \Omega, \tau > \tau_-, s_i \geq (s_i)_- \text{ for } i \in [1, N], s_{i_0} > (s_{i_0})_- \text{ for some } i_0 \in [1, N]\}, \tag{1.22}$$

while it cannot contain a critical point. The conclusion will directly follow since by definition $\mathcal{O}^+$ enters this region. Indeed since by assumption $\mu := \sum_{i=1}^N \mu_i > 0$, there exists at least one index $i_0 \in [1, N]$ with $\mu_{i_0} > 0$ so that $d_\xi s_{i_0} > 0$, that is to say $s_{i_0}(\xi) > (s_{i_0})_-$.
The required properties are an easy consequence of the next estimate :

$$\mathcal{F}(\tau, \{s_i\}_{i=1,\ldots,N}) > 0, \text{ for all } \mathbf{u} \in \mathcal{D}.$$

Let us first observe that the mapping $\mathcal{F}(\tau, \{s_i\}_{i=1,\ldots,N})$ strictly increases with each of the specific entropies ($\partial_{s_i}\mathcal{F} = \tau^{-\gamma_i}$), hence any state $\mathbf{u} \in \mathcal{D}$ (*i.e* with $s_i > (s_{i_0})_-$ for some $i_0 \in [1, N]$) obeys :

$$\mathcal{F}(\tau, \{(s_i)_-\}_{i=1,\ldots,N}) < \mathcal{F}(\tau, \{s_i\}_{i=1,\ldots,N}).$$

Let us then conclude when establishing that

$$0 = \mathcal{F}(\tau_-, \{(s_i)_-\}_{i=1,\ldots,N}) < \mathcal{F}(\tau, \{(s_i)_-\}_{i=1,\ldots,N}) \text{ for all } \tau > \tau_-. \tag{1.23}$$

This estimate follows when observing that the mapping $\mathcal{F}(\tau, \{s_i\}_{i=1,\ldots,N})$ is strictly convex in the τ variable, therefore for all $\tau > \tau_-$:

$$\partial_\tau \mathcal{F}(\tau, \{(s_i)_-\}_{i=1,\ldots,N}) > \partial_\tau \mathcal{F}(\tau_-, \{(s_i)_-\}_{i=1,\ldots,N}) = (\rho c)_-^2 ((\frac{m}{\rho c})_-^2 - 1).$$

But since the relative mach number must meet the condition (1.17), the mapping $\mathcal{F}(\tau, \{(s_i)_-\}_{i=1,\ldots,N})$ strictly increases with τ and (1.23) immediately follows. These considerations therefore preclude $\mathcal{O}^+$ to connect some critical point in the future. The orbit $\mathcal{O}^-$ stays the only possible candidate which we now prove to be necessarily compressive in its maximal interval of existence. Indeed and by definition, $\mathcal{O}^-$ is necessarily made of states with $\tau < \tau_-$ and $\mathcal{F}(\tau, \{(s_i)\}_{i=1,\ldots,N}) < 0$ for $|\tau - \tau_-|$ sufficiently small (in view of $\partial_\tau \mathcal{F}(\tau_-, \{(s_i)_-\}_{i=1,\ldots,N}) > 0$, see (1.18)). Let us then observe that the mapping $\mathcal{F}$ stays necessarily negative for all values of ξ smaller than the maximal time of existence of the orbit $\mathcal{O}^-$. Indeed, states at which the mapping $\mathcal{F}$ vanishes are by definition critical points of the dynamical system (1.12) and as it is well-known, such states cannot be reached in finite time. This concludes the proof.

In view of Proposition 3, the application $\xi \to \tau(\xi)$ maps the maximal interval of existence $]-\infty, \xi^+_{max}[$ of the orbit connecting $\mathbf{u}_-$ in the past onto $]\tau_{min}, \tau_-[$ for some $\tau_{min} \geq 0$. Its strict monotonicity allows for defining the inverse mapping $\tau \to \xi(\tau)$, from $]\tau_{min}, \tau_-[$ onto $]-\infty, \xi^+_{max}[$ and thus for understanding any given function of ξ as a function of the covolume τ. Specializing this property to each of the specific entropies s_i with $i \in [1, N]$, the N mappings $\tau \in]\tau_{min}, \tau_-[\to s_i(\tau)$ defined from the orbit $\mathcal{O}^-$ of the dynamical system (1.12) are easily seen to obey the following linear ODE system with variable coefficients:

$$d_\tau s_i = (\gamma_i - 1)\tau^{(\gamma_i - 1)}\frac{\mu_i}{\mu}\mathcal{F}(\tau, \{s_i\}_{i=1,...,N}), \quad i = 1, ..., N, \tag{1.24}$$

where τ plays the role of a time like variable. Notice that the above system stays meaningfull when evaluated at time $\tau = \tau_-$. Let us thus study for times $\tau \leq \tau_-$ the existence of a solution of this non autonomous system with initial data $s_i(\tau_-) = (s_i)_-$, $i = 1, ..., N$ without reference to the time τ_{min} (*i.e.* without reference to the possibly finite maximal time ξ^+_{max} of existence of $\mathcal{O}^-$). Let us indeed observe that the solution under consideration exists for all positive time τ since the coefficients of (1.24) are then Lipschitz-continuous. Our purpose is to prove for existence a first positive time $\tau_\star < \tau_-$ such that

$$\mathcal{F}(\tau_\star, \{s_i(\tau_\star)\}_{i=1,...,N}) = 0, \tag{1.25}$$

with the property that the solutions $\{s_i(\tau)\}_{i=1,...,N}$ stays larger than $\{s_i(\tau_-)\}_{i=1,...,N}$ for all $\tau \in [\tau_\star, \tau_-]$. Observe that the algebraic equation (1.25) expresses nothing but the conservation of the total impulsion. The existence of such a time $\tau_\star > 0$ will be seen to imply that the orbit $\mathcal{O}^-$ necessarily connects in the future a critical point of the dynamical system (1.12): namely $\mathbf{u}_+ := (\tau_\star, \{s_i(\tau_\star)\}_{i=1,...,N})$.

1.3.2 Reduction to an algebraic problem

We prove below that the solution we seek for, can be actually inferred from the solution of a linear ODE problem with constant coefficients. Equipped with its solution explicitly given by the Duhamel representation formula, we shall then prove that the required time $\tau_\star$ exists and just solves a non linear algebraic equation equivalent to (1.25). More precisely, each of the specific entropies s_i will be rebuilt from the following dimensionless variables:

$$\theta_i = \frac{1}{(\gamma_i - 1)}\frac{(p_i - (p_i)_-)}{m^2\tau_-}, \quad i = 1, ..., N, \tag{1.26}$$

together with:

$$x = -\ln(\frac{\tau}{\tau_-}). \tag{1.27}$$

Observe that $\theta_i = 0$ when $p_i = (p_i)_-$ while $x \geq 0$ when $\tau \leq \tau_-$. x will play the role of a dimensionless time like variable while the dimensionless vector $\mathbf{\Theta} = \{\theta_i\}_{i=1,...,N}$

will denote the solution of the following linear ODE system with constant coefficients derived from the non autonomous system (1.24) at the end of the present section:

$$\begin{cases} d_x\mathbf{\Theta} - \mathcal{M}\mathbf{\Theta} = \alpha\mathbf{a} + (1-\exp(-x))\mathbf{b}, \\ \mathbf{\Theta}(x=0) = 0. \end{cases} \tag{1.28}$$

Here, the constant vectors of $I\!R^N$ entering the right hand side read:

$$\mathbf{a} = {}^T(\{\beta_i/(\gamma_i - 1)\}_{i=1,\dots,N}), \quad \mathbf{b} = {}^T(\{\mu_i/\mu\}_{i=1,\dots,N}), \tag{1.29}$$

where the dimensionless coefficients β_i with $i \in \{1,...,N\}$ stand for the following ratios of sound speeds:

$$\beta_i = \frac{\gamma_i(p_i\tau)_-}{c_-^2}, \quad i = 1,...,N, \quad \alpha = (\frac{(\rho c)_-}{m})^2. \tag{1.30}$$

α clearly coincides with the squarred inverse of the relative Mach number. Next in (1.28), $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1,\dots,N}$ stands for a $N \times N$ matrix with constant real coefficients given by:

$$\mathcal{M}_{ij} = \gamma_i\delta_{ij} - \mu_i(\gamma_j - 1)/\mu, \tag{1.31}$$

where δ denotes the Kronecker symbol. We shall briefly report some interesting properties of this matrix later on, together with some comments about the role of the dimensionless coefficients involved in (1.28). Let us just underline that in our forthcoming developments, the reduced number α will serve as a natural parameter in the solution of (1.28) to represent all the possible speeds of propagation σ that satisfy the Lax inequality (1.17), the left state $\mathbf{u}_-$ being tacitly fixed. As a consequence, α will belong to $[0,1]$.

Let us now turn deriving the algebraic problem when recalling that the solution of the linear ODE problem (1.28) obeys the Duhamel formula:

$$\mathbf{\Theta}(x) = \alpha\int_0^x \exp[\mathcal{M}(x-y)]dy\ \mathbf{a} + \int_0^x [1-\exp(-y)]\exp[\mathcal{M}(x-y)]dy\ \mathbf{b}. \tag{1.32}$$

On the ground of this solution, we define the mapping $x \to \mathbf{u}(x)$ setting for $x > 0$:

$$\begin{gathered} \tau(x) = \tau_-\exp(-x), \quad u(x) = u_- - m\tau_-(1-\exp(-x)), \\ p_i(x) = (p_i)_- + m^2\tau_-(\gamma_i - 1)\theta_i(x), \quad i = 1,...,N. \end{gathered} \tag{1.33}$$

Equipped with this mapping (which we do not claim to belong to the phase space Ω for all x), the next issue is concerned with the existence of positive roots x of the following algebraic problem:

$$I(x) = \sum_{i=1}^N p_i(x) + m^2\tau(x) = I_- := \sum_{i=1}^N (p_i)_- + m^2\tau_-. \tag{1.34}$$

The smallest positive root, which we denote $x(\alpha)$, will naturally serve to provide the existence of a critical point $\mathbf{u}_+$ which can be connected by the orbit $\mathcal{O}^-$ in the future, as soon as $\mathbf{u}(x) \in \Omega$ for all $x \in [0, x(\alpha)]$. Indeed, observe that $\tau_\star = \tau(x(\alpha))$ is nothing but the value entering the equation (1.25). This problematic reexpresses equivalently in terms of the dimensionless variables $\{\theta_i\}_{i=1,...,N}$ as follows: prove the existence of a smallest positive root to:

$$\sum_{i=1}^{N}(\gamma_i - 1)\theta_i(x) - (1 - \exp(-x)) = 0. \tag{1.35}$$

Taking advantage of the Duhamel representation formula (1.32), the above algebraic equation just recasts:

$$\psi(x,\alpha) := \alpha < \kappa, \int_0^x \exp[\mathcal{M}(x-y)]dy\ \mathbf{a} >$$
$$+ < \kappa, \int_0^x [1-\exp(-y)]\exp[\mathcal{M}(x-y)]dy\ \mathbf{b} > -(1-\exp(-x)) = 0,$$

where we have introduced $\kappa = {}^T(\{\gamma_i - 1\}_{i-1,...,N})$. Let us recall that the parameter α belongs to $[0,1]$. Then, setting

$$\phi(x) =< \kappa, \int_0^x \exp[\mathcal{M}(x-y)]dy\ \mathbf{a} >,$$

we have to prove the existence of positive roots of:

$$\psi(x,\alpha) = \alpha\phi(x) + \psi(x,0) = 0, \tag{1.36}$$

and select the smallest one, say $x(\alpha)$. This will be the matter of the next section.

Lemma 2 *Let be given a fixed state $\mathbf{u}_- \in \Omega$ and the associated reduced numbers $(\beta_i)_{i=1,...,N}$ in $[0,1]^N$ with $\sum_{i=1}^N \beta_i = 1$. Then assume that for α in $[0,1]$, the non linear algebraic problem (1.35) admits at least one positive solution. Denoting $x(\alpha)$ the smallest positive root, assume in addition that:*

$$\theta_i(x) \geq 0 \textit{ for all } x \in [0, x(\alpha)] \textit{ and all } i = 1,...,N. \tag{1.37}$$

Then the orbit $\mathcal{O}^-$ connecting $\mathbf{u}_-$ in the past, connects a state $\mathbf{u}_+$ in future times given by:

$$\begin{aligned} &\tau_+ = \tau_- \exp(-x(\alpha)), \quad u_+ = u_- - m\tau_-(1-\exp(-x(\alpha))), \\ &(p_i)_+ = (p_i)_- + m^2\tau_-(\gamma_i - 1)\theta_i(x(\alpha),\alpha), \quad i = 1,...,N. \end{aligned} \tag{1.38}$$

The proof of this result is postponed at the end of the present section.

Let us observe that the dimensionless system (1.28) and the algebraic equation are kept unchanged when reversing the sign of the relative Mach number m. This invariance property thus ensures that an analysis of the travelling wave solutions associated with the third field will result in exactly the same reduced problem (provided that the role of the exit states $\mathbf{u}_-$ and $\mathbf{u}_+$ are exchanged). In other words, solving in $x(\alpha,\beta)$ the non linear equation (1.36) for all $\alpha \in [0,1]$ and all dimensionless vector β in the following subset of $I\!R^N$:

$$\mathbf{B} = \{\beta \in I\!R^N /\ \beta_i \geq 0,\ \ 1 \leq i \leq N,\quad \sum_{i=1}^N \beta_i = 1\}, \tag{1.39}$$

will suffice to generate a complete family of travelling waves for either the first or the third non linear field. Let us emphasize that these travelling wave solutions heavily depend on the N viscosity ratios μ_i/μ, since does the matrix $\mathcal{M}$ and the vector $\mathbf{b}$ in (1.28). Let us just mention that whatever are these ratios, the matrix $\mathcal{M}$ is a diagonally dominant M-matrix (*i.e.* with positive diagonal entries and non positive extra diagonal terms).

Let us now prove Lemma 2.

Proof For any given x in $]0, x(\alpha)[$, let us define a state $\mathbf{u}(x)$ in Ω when considering under the assumption (1.37) :

$$\tau(x) = \tau_- \exp(-x), \quad u(x) = u_- - m\tau_-(1 - \exp(-x)),$$

$$p_i(x) = (p_i)_- + m^2\tau_-(\gamma_i - 1)\theta_i(x) \geq (p_i)_-, \quad i = 1, ..., N.$$

Such a family of states can obviously be reparametrized in terms of the covolume and by construction the mapping $\tau \to \mathbf{u}(\tau)$, $\tau \in]\tau_- \exp(-x(\alpha)), \tau_-[$, is nothing but the solution of (1.24) with $s_i(\tau_-) = (s_i)_-$ for all $i = 1, ..., N$. Notice that for all $\tau \in]\tau_- \exp(-x(\alpha)), \tau_-[$, we have $\mathcal{F}(\tau, \{s_i(\tau)\}_{i=1,...,N}) < 0$ while

$$\mathcal{F}(\tau_- \exp(-x(\alpha)), \{s_i(\tau_- \exp(-x(\alpha)))\}_{i=1,...,N}) = 0.$$

Then, for all $\tau \in]\tau_- \exp(-x(\alpha)), \tau_-[$,

$$\frac{d\xi}{d\tau} = \frac{\mu m}{\mathcal{F}(\tau, \{s_i(\tau)\}_{i=1,...,N})} < 0,$$

and the well-defined mapping $\xi \to \tau(\xi)$ maps $]\tau_+, \tau_-[$ on $]-\infty, \xi^+_{max}[$ with $\xi^+_{max} \in \overline{I\!R}$. Then for all $\xi \in]-\infty, \xi^+_{max}[$, we have defined a mapping $\xi \to \mathbf{u}(\xi)$ as the solution of the dynamical system (1.12). But using the fact that $\mathcal{F}(\tau_+, \{s_i(\tau_+)\}_{i=1,...,N}) = 0$, then necessarily $\xi^+_{max} = +\infty$. This concludes the proof.

Let us now conclude this section when deriving the linear ODE system from the non autonomous system (1.24).

Proof By virtue of the second principle of the thermodynamics, we have :

$$\frac{\tau^{(1-\gamma_i)}}{(\gamma_i - 1)} ds_i = d\varepsilon_i + p_i d\tau, \quad i = 1, ..., N,$$

so that (1.24) is easily seen to reexpress equivalently as :

$$d_\tau \varepsilon_i + p_i = \frac{\mu_i}{\mu}\mathcal{F}(\tau, \{s_i\}_{i=1,...,N}), \quad i = 1, ..., N.$$

But using the definition of $\mathcal{F}$ we get :

$$d_\tau \varepsilon_i + p_i = \frac{\mu_i}{\mu}(m^2(\tau - \tau_-) + (p - p_-)), \quad i = 1, ..., N. \tag{1.40}$$

Consequently, (1.40) reads, taking advantage of the identities $\varepsilon_i = p_i\tau/(\gamma_i - 1)$ for all $i = 1, ..., N$:

$$\frac{1}{\gamma_i - 1}\tau d_\tau p_i + \frac{\gamma_i}{\gamma_i - 1}p_i - \frac{\mu_i}{\mu}\sum_{j=1}^{N} p_j = -\frac{\mu_i}{\mu}\sum_{j=1}^{N}(p_j)_- + \frac{\mu_i}{\mu}m^2(\tau - \tau_-). \tag{1.41}$$

These equations then strongly suggest to consider the admissible change of variable $x = -\ln(\frac{\tau}{\tau_-})$ so as to recast (1.41) under the form of a linear system of ODE in the x variable :

$$\frac{1}{\gamma_i - 1}d_x p_i - \frac{\gamma_i}{\gamma_i - 1}p_i + \frac{\mu_i}{\mu}\sum_{j=1}^{N} p_j = \frac{\mu_i}{\mu}\sum_{j=1}^{N}(p_j)_- - \frac{\mu_i}{\mu}m^2\tau_- \exp(-x)(1 - \exp(x)).$$

Next invoking the definition (1.26) of the dimensionless variables $\{\theta_i\}_{i=1,...,N}$, we easily get the equivalent form :

$$\begin{cases} d_x\theta_i - \gamma_i\theta_i + \frac{\mu_i}{\mu}\sum_{j=1}^{N}(\gamma_j - 1)\theta_j = \alpha\frac{\beta_i}{\gamma_i - 1} + \frac{\mu_i}{\mu}(1 - \exp(-x)), \quad i = 1, ..., N, \\ \theta_i(x = 0) = 0, \end{cases} \tag{1.42}$$

which is nothing but the linear ODE problem (1.28).

1.3.3 Solvability of the algebraic problem

In this section we prove that the algebraic problem (1.36) can be solved for all $\alpha \in [0, 1]$. Notice that for all these values of α, $x = 0$ is always solution. But as soon as $\alpha < 1$, Proposition 3 gives the existence of values of the covolume $\tau(\xi) < \tau_-$ for ξ sufficiently negative. Since the orbit $\mathcal{O}^-$ has been proved to be necessarily compressive, this obviously discards $x = 0$ (*i.e.* $\tau(0) = \tau_-$) to be the solution of interest when $\alpha < 1$. By contrast, $x = 0$ is expected to be the admissible solution when considering $\alpha = 1$. Depending on the left state $\mathbf{u}_-$, the algebraic problem (1.36) may admit up to two distinct positive solutions. According to Lemma 2, we are led to select the smallest positive one, say $x(\alpha)$, when $\alpha < 1$ and we prove below that this selection procedure satisfies the positiveness assumption (1.37) stated in Lemma 2. We shall have thus proved that for any given state $\mathbf{u}_-$ and speed σ prescribed according to (1.17), there exists an unique heteroclinic solution, namely $\mathcal{O}^-$, connecting in the future the state $\mathbf{u}_+$ given in (1.38).

Proposition 4 *Let be given $\alpha \in [0, 1[$, then there exists at least one positive root of $\psi(x, \alpha) = 0$. Denoting $x(\alpha)$ the smallest positive root, we have for all $x \in [0, x(\alpha)]$:*

$$\theta_i(x, \alpha) \geq 0, \quad i = 1, ..., N. \tag{1.43}$$

The next statement proves that the above derivation principle allows for defining a continuous function $\alpha \to x(\alpha)$ on the interval $[0, 1]$:

Proposition 5 *The function $\alpha \to x(\alpha)$ satisfies the following transversality considition :*

$$\{\partial_x\psi\}(x(\alpha), \alpha) > 0, \quad \alpha \in [0, 1). \tag{1.44}$$

The mapping $\alpha \to x(\alpha)$ is thus continuous on the interval $[0, 1[$ with moreover

$$\lim_{\alpha \to 1^-} x(\alpha) = 0.$$

Let us first prove Proposition 4.

Proof Let us first fix $\alpha \in [0,1[$ and prove that $\frac{\partial\psi}{\partial x}(0,\alpha) < 0$. Since $\psi(0,\alpha) = 0$, we shall have $\psi(x,\alpha) < 0$ for $x > 0$, x sufficiently small. We shall then establish that $\psi(x,\alpha)$ cannot keep negative values for all $x > 0$. This property will thus prove the existence of a first positive root of $\psi(x,\alpha) = 0$.
The proof heavily relies on the observation that the ODE system (1.42) naturally writes as follows:

$$\frac{d\theta_i}{dx} - \gamma_i\theta_i = \alpha\frac{\beta_i}{\gamma_i - 1} - \frac{\mu_i}{\mu}\psi(x,\alpha), \quad i = 1, ..., N. \tag{1.45}$$

Let us thus begin by proving that $\frac{\partial\psi}{\partial x}(0,\alpha) < 0$. A direct calculation gives:

$$\frac{\partial\psi}{\partial x}(0,\alpha) = \sum_{i=1}^{N}(\gamma_i - 1)\frac{\partial\theta_i}{\partial x}(0,\alpha) - 1,$$

so that, in view of (1.45) and since $\psi(0,\alpha) = 0$, $\theta_i(0,\alpha) = 0$ for all $i = 1, ..., N$, we have:

$$\frac{\partial\theta_i}{\partial x}(0,\alpha) = \alpha\frac{\beta_i}{\gamma_i - 1},$$

with little abuse in the notations. As a consequence, since $\sum_{i=1}^{N}\beta_i = 1$, we get the required estimate for all $0 \leq \alpha < 1$:

$$\frac{\partial\psi}{\partial x}(0,\alpha) = \alpha(\sum_{i=1}^{N}\beta_i) - 1 = \alpha - 1 < 0.$$

Next, in order to prove that the mapping $x \to \psi(x,\alpha)$ admits at least one positive root for all $0 \leq \alpha < 1$, let us assume that $\psi(x,\alpha) < 0$ for all $x \in I\!R^{+,\star}$ to finally rise a contradiction. Under this assumption, the equivalent form (1.45) of the ODE system immediately implies the following inequality for all $i = 1, ..., N$:

$$\frac{d\theta_i}{dx} - \gamma_i\theta_i > \alpha\frac{\beta_i}{\gamma_i - 1},$$

so that we get:

$$\frac{d}{dx}(\theta_i\exp(-\gamma_i x)) > \alpha\frac{\beta_i}{\gamma_i - 1}\exp(-\gamma_i x).$$

Integrating from 0 to x thus yields the positiveness properties:

$$\theta_i(x,\alpha) > \alpha\frac{\beta_i}{\gamma_i(\gamma_i - 1)}(\exp(\gamma_i x) - 1) \geq 0, \tag{1.46}$$

for all $\alpha \in [0,1)$. To rise the contradiction for all $\alpha \in [0,1)$ (including $\alpha = 0$), let us establish the next identity:

$$\sum_{i=1}^{N}\theta_i(x,\alpha) = \exp(x)[1 - \exp(-x)][\alpha\sum_{i=1}^{N}(\frac{\beta_i}{\gamma_i - 1}) + \frac{1}{2}(1 - \exp(-x))]. \tag{1.47}$$

This follows by summing the N equations in (1.45) to obtain the ODE:

$$\frac{d}{dx}(\sum_{i=1}^{N}\theta_i) - (\sum_{i=1}^{N}\theta_i) = \alpha\sum_{i=1}^{N}\frac{\beta_i}{\gamma_i - 1} + (1-\exp(-x)),$$

which once integrated from 0 to x gives (1.47). Then, the positiveness property (1.46) of each of the θ_i easily implies the following inequality:

$$(\gamma_m - 1)\sum_{i=1}^{N}\theta_i(x,\alpha) \le \sum_{i=1}^{N}(\gamma_i - 1)\theta_i(x,\alpha), \tag{1.48}$$

where $\gamma_m = \min_{i=1,\dots,N}\gamma_i > 1$. Invoking (1.47), this inequality implies the following lower bound, valid for all $\alpha \in [0,1)$:

$$\begin{aligned}\tfrac{1}{2}[1-\exp(-x)][2\alpha(\gamma_m - 1)\sum_{i=1}^{N}(\tfrac{\beta_i}{\gamma_i - 1})\exp(x) + (\gamma_m - 1)\exp(x) - (\gamma_m + 1)]&\\ \le \psi(x,\alpha).&\end{aligned} \tag{1.49}$$

To finally rise the contradiction, it suffices to notice that the left hand side in (1.49) becomes positive for x sufficiently large. As a consequence, there exists necessarily at least one positive root and the function $\alpha \to x(\alpha)$ is well-defined for all $\alpha \in [0,1)$.
To conclude, let us choose $\alpha \in [0,1)$ so that for all x in the non empty interval $(0, x(\alpha))$, we have by construction $\psi(x,\alpha) < 0$. Inequalities (1.46) are still valid and yield the expected estimates (1.43).

We now establish the required continuity property.

Proof Let us first derive the transversality result (1.44). In that aim, let us observe that the function $x \to \psi(x,\alpha)$ is $\mathcal{C}^\infty(I\!R^+)$ since is $x \to \theta(x,\alpha)$, and let us evaluate the x partial derivative of this mapping on $x(\alpha)$:

$$\frac{\partial\psi}{\partial x}(x(\alpha),\alpha) = \sum_{i=1}^{N}(\gamma_i - 1)\frac{\partial\theta_i}{\partial x}(x(\alpha),\alpha) - \exp(-x(\alpha)).$$

Let us then recall that the ODE system (1.42) reads equivalently:

$$(\frac{d\theta_i}{dx} - \gamma_i\theta_i)(x(\alpha)) = \alpha\frac{\beta_i}{\gamma_i - 1} - \frac{\mu_i}{\mu}\psi(x(\alpha),\alpha),$$

so that, with a little abuse in the notations:

$$\frac{\partial\theta_i}{\partial x}(x(\alpha),\alpha) = \alpha\frac{\beta_i}{\gamma_i - 1} + \gamma_i\theta_i(x(\alpha),\alpha), \quad i = 1,\dots,N.$$

Invoking now the definitions (1.26) and (1.30) for θ_i and β_i, $i = 1,\dots,N$, we have:

$$\begin{aligned}\frac{\partial\psi}{\partial x}(x(\alpha),\alpha) &= \alpha(\sum_{i=1}^{N}\beta_i) + \sum_{i=1}^{N}\gamma_i(\gamma_i - 1)\theta_i(x(\alpha),\alpha) - \exp(-x(\alpha))\\ &= \sum_{i=1}^{N}\frac{\gamma_i p_i(x(\alpha),\alpha)}{m^2\tau_-} - \exp(-x(\alpha)) + (\alpha - \frac{c_-^2}{m^2\tau_-^2}),\end{aligned}$$

where the last term vanishes by the definition of α. Using the identity $\exp(-x(\alpha)) = \tau(\alpha)/\tau_-$, we arrive at the following convenient form:

$$\frac{\partial\psi}{\partial x}(x(\alpha),\alpha) = \exp(-x(\alpha))(\{\frac{\sum_{i=1}^N \gamma_i p_i}{m^2\tau}\}(x(\alpha),\alpha) - 1).$$

But on $x(\alpha)$, that is to say on the exit state $\mathbf{u}_+(\alpha)$ of the heteroclinic solution connecting $\mathbf{u}_-$ in the past for the relative mach number $\alpha^{-1/2}$, we necessarily have:

$$\{\frac{m^2\tau}{\sum_{i=1}^N \gamma_i p_i}\}(x(\alpha),\alpha) = \frac{m^2}{(\rho c)_+^2} < 1,$$

as soon as $\alpha < 1$. This estimate is nothing but the Lax inequality already put forward in (1.19). We have therefore proved the required inequality: namely $\frac{\partial\psi}{\partial x}(x(\alpha),\alpha) > 0$ for all $\alpha \in [0,1)$. Noticing that the mapping $(x,\alpha) \in I\!R^+ \times [0,1) \to \psi(x,\alpha)$ achieves the same smoothness as the mappings $\theta_i(x,\alpha)$, the function $\alpha \to x(\alpha)$ is necessarily continuous on $[0,1)$ in view of the transversality result (1.44).
Let us next prove that this function admits a limit as α goes to 1, and that this limit is necessarily null. In that aim, let be given $\alpha < 1$ so that for all $x \in [0, x(\alpha)]$, $\psi(x,\alpha) \leq 0$. For all these values of x, we observe that the estimates (1.46) are valid, so that the next lower bound holds:

$$g(x,\alpha) := \alpha \sum_{i=1}^N \frac{\beta_i}{\gamma_i}(\exp(\gamma_i x) - 1) - (1 - \exp(-x)) \leq \psi(x,\alpha). \tag{1.50}$$

But, direct calculations prove that the function $g(x,\alpha)$ in strictly convex in the x variable ($\alpha > 0$) with $g(0,\alpha) = 0$, while $\partial_x g(0,\alpha) = \alpha - 1 \leq 0$. As a consequence, this function admits exactly two zero located in 0 and $\overline{x}(\alpha)$. Clearly, $\alpha \to \overline{x}(\alpha)$ is continuous on $(0,1]$ with $\lim\limits_{\alpha\to 1^-} \overline{x}(\alpha) = 0$. Indeed, observe that $\partial_x g(0,1) = 0$. Equipped with this auxiliary function, we necessarily deduce from the estimate (1.50) and the definition of $x(\alpha)$ (*i.e.* $\psi(x,\alpha) < 0$ for $x \in (0, x(\alpha))$) the following inequality:

$$x(\alpha) \leq \overline{x}(\alpha), \quad \text{for all } \alpha \in (0,1].$$

This completes the proof.

1.3.4 Additional properties

This section aims at establishing further properties of the function $x(\alpha)$ for $\alpha \in [0,1]$, which will be of primary importance in the forthcoming analysis of the Riemann problem. All the proofs are postponed at the end of this section. Here, the main result is:

Proposition 6 *The function $\alpha \to x(\alpha)$ is continuously differentiable on $(0,1)$ and decreases in the large. More precisely, there exists α_c, $0 \leq \alpha_c < 1$, such that $x(\alpha)$ is strictly decreasing on $]\alpha_c, 1[$ while necessarily $x(\alpha) = x(0)$ for all $0 \leq \alpha \leq \alpha_c$.*

In other words, the compression rate τ_-/τ_+ grows monotonically with the relative Mach number. One would have expected a strict monotonicity property but existence and uniqueness of the Riemann solution will follow without the need for proving that

$\alpha_c = 0$. This might not seem too surprising since Smith has proved uniqueness in the setting $N = 1$ provided that the compression rate does not depart too much from monotonicity (see the medium condition [129]). The next result states that the compression rate stays bounded in the regime of an infinite relative Mach number ($\alpha = 0$). More precisely :

Lemma 3 *Let us define* $\gamma_m = \min_{i=1,...,N} \gamma_i$ *and* $\gamma_M = \max_{i=1,...,N} \gamma_i$. *Then, the mapping* $\alpha \to x(\alpha)$ *obeys the following estimates on the maximal compression rate :*

$$\frac{\gamma_m - 1}{\gamma_m + 1} \leq \min_{\alpha \in [0,1]} \exp(-x(\alpha)) = \exp(-x(0)) \leq \frac{\gamma_M - 1}{\gamma_M + 1}. \tag{1.51}$$

Let us notice that the bounds (1.51) are optimal since when considering identical adiabatic exponents, *i.e.* $\gamma = \gamma_i$ for all $i = 1, ..., N$, these inequalities express nothing but the identity :

$$\exp(-x(0)) = \frac{\gamma - 1}{\gamma + 1}, \tag{1.52}$$

which is well-known in the usual setting of a single pressure law ($N = 1$). The proof of Lemma 3 will easily yield under the assumption of N identical adiabatic exponents :

$$\exp(-x(\alpha)) = \frac{\gamma - 1}{\gamma + 1}(1 + 2\alpha \sum_{i=1}^{N} \frac{\beta_i}{\gamma_i - 1}),$$

(see indeed (1.58) below), so that in this simplified setting, $\alpha \to x(\alpha)$ is strictly decreasing. Such a property naturally suggests the last result of this section :

Proposition 7 *Let us assume that* $(\gamma_M - \gamma_m)$ *is sufficiently small. Then, the mapping* $\alpha \to x(\alpha)$ *is strictly decreasing on* $[0, 1]$.

We now prove Proposition 6.

Proof Let us first observe that the mapping $\alpha \to x(\alpha)$ is smooth on $(0, 1)$ as a by product of the implicit functions theorem. Indeed the transversality result (1.44) we have just proved shows that $\frac{\partial \psi}{\partial x}(x, \alpha) > 0$ for $|x - x(\alpha)|$ sufficiently small and $\alpha \in (0, 1)$.

Let us now turn establishing the expected monotonicity property. The proof will rely on the equivalent formulation (1.36) of the algebraic problem to be solved, namely :

$$\psi(x, \alpha) = \alpha\phi(x) + \psi(x, 0) = 0. \tag{1.53}$$

The smoothness of the function $\alpha \to \psi(x(\alpha), \alpha)$ for $\alpha \in (0, 1)$ allows to consider :

$$\begin{aligned} \frac{d}{d\alpha}\psi(x(\alpha), \alpha) &= \phi(x(\alpha)) + \frac{\partial \psi}{\partial x}(x(\alpha), \alpha) \times x'(\alpha) \\ &= -\frac{1}{\alpha}\psi(x(\alpha), 0) + \frac{\partial \psi}{\partial x}(x(\alpha), \alpha) \times x'(\alpha), \end{aligned}$$

since by definition, $x(\alpha)$ solves (1.53). We therefore get for all $\alpha \in (0, 1)$:

$$\frac{\partial \psi}{\partial x}(x(\alpha), \alpha) \times x'(\alpha) = \frac{1}{\alpha}\psi(x(\alpha), 0). \tag{1.54}$$

Let us recall that $\frac{\partial\psi}{\partial x}(x(\alpha),\alpha)>0$. As a consequence, the expected decreasing property holds for all α such that $x(\alpha)<x(0)$ since we necessarily have $\psi(x(\alpha),0)<0$. Indeed, recall that $x(0)$ is the first positive root of $\psi(x,0)=0$. Let us now prove that there exists an non empty interval $(\alpha_c,1)$ where $x(\alpha)<x(0)$. The existence of such a non trivial interval follows from $x(0)>0=x(1)$ and the continuity of the function $\alpha\to x(\alpha)$ on $[0,1]$. Let us then consider the largest open interval $(\alpha_c,1)$ where $x(\alpha)<x(0)$, *i.e.* with $x(\alpha_c)=x(0)$. We have just proved that the function $\alpha\to x(\alpha)$ strictly decreases on this maximal interval and maps $[\alpha_c,1]$ onto $[x(1)=0,x(0)]$.
Let us assume that $\alpha_c>0$ and prove that necessarily $x(\alpha)=x(0)$ for all $\alpha\in[0,\alpha_c]$. In that aim, observe first that the identity $x(\alpha_c)=x(0)$ implies, since $x(\alpha_c)$ solves the algebraic problem (1.53) :

$$\psi(x(\alpha_c),\alpha_c)=\alpha_c\phi(x(0))+\psi(x(0),0)=0. \tag{1.55}$$

But by construction, $\psi(x(0),0)=0$ so that $\alpha_c>0$ gives in addition that $\phi(x(0))=0$. These two identities necessarily make $x(0)$ to be a positive solution of the algebraic problem (1.53) for any given value of $\alpha\in(0,1)$. We therefore infer that :

$$x(\alpha)\leq x(0),\quad \text{for all } \alpha\in(0,\alpha_c), \tag{1.56}$$

since again by construction $x(\alpha)$ is the first positive solution of (1.53). To conclude, let us assume $x(\alpha)<x(0)$ for some given $\alpha\in(0,\alpha_c)$ to rise a contradiction. To this purpose, recall that the function $\alpha\to x(\alpha)$ have been shown to map $[\alpha_c,1]$ onto $[x(1)=0,x(0)]$, so that its continuity proves for existence $\underline{\alpha}\in(\alpha_c,1)$ such that $x(\underline{\alpha})=x(\alpha)$. Let us then prove that necessarily $\phi(x(\underline{\alpha}))=0$. This follows from :

$$\psi(x(\underline{\alpha}),\underline{\alpha})=\underline{\alpha}\phi(x(\underline{\alpha}))+\psi(x(\underline{\alpha}),0)=\alpha\phi(x(\underline{\alpha}))+\psi(x(\underline{\alpha}),0)=\psi(x(\alpha),\alpha)=0,$$

with $\alpha<\underline{\alpha}$. But $\phi(x(\underline{\alpha}))=0$ implies in turn that $\psi(x(\underline{\alpha}),0)=0$. Therefore $x(\underline{\alpha})$ solves $\phi(x,0)=0$ so that $x(\underline{\alpha})>x(0)$ since $x(0)$ is the first positive root. Hence we would arrive at $x(\alpha)=x(\underline{\alpha})>x(0)$ and this estimate is in contradiction with (1.56). We have just proved that necessarily $x(\alpha)=x(0)$ for all $\alpha\in[0,\alpha_c]$. This concludes the proof.

Let us then prove Lemma 3.

Proof Let us choose $\alpha\in[0,1[$. Arguments, identical to those developed in the proof of Proposition 4, ensure for validity the estimates (1.48) et (1.49) for all $x\in[0,x(\alpha)]$. Plugging $x(\alpha)$ in (1.49) yields :

$$\begin{aligned}\tfrac{1}{2}[1-\exp(-x(\alpha))][2\alpha(\gamma_m-1)\sum_{i=1}^{N}(\tfrac{\beta_i}{\gamma_i-1})\exp(x(\alpha))+(\gamma_m-1)\exp(x(\alpha))-(\gamma_m+1)]\\ \leq\psi(x(\alpha),\alpha)=0.\end{aligned} \tag{1.57}$$

Symmetrical steps apply to prove the next inequality :

$$\begin{aligned}\psi(x(\alpha),\alpha)=0\leq\\ \tfrac{1}{2}[1-\exp(-x(\alpha))][2\alpha(\gamma_M-1)\sum_{i=1}^{N}(\tfrac{\beta_i}{\gamma_i-1})\exp(x(\alpha))+(\gamma_M-1)\exp(x(\alpha))-(\gamma_M+1),]\end{aligned}$$

where have set $\gamma_M = \max_{i=1,\dots,N} \gamma_i$. These inequalities easily yield the following rough estimates :

$$\frac{\gamma_m - 1}{\gamma_m + 1}(1 + 2\alpha \sum_{i=1}^{N} \frac{\beta_i}{\gamma_i - 1}) \leq \exp(-x(\alpha)) \leq \frac{\gamma_M - 1}{\gamma_M + 1}(1 + 2\alpha \sum_{i=1}^{N} \frac{\beta_i}{\gamma_i - 1}), \tag{1.58}$$

so that we have :

$$\frac{\gamma_m - 1}{\gamma_m + 1} \leq \min_{\alpha \in [0,1]} \exp(-x(\alpha)) \leq \frac{\gamma_M - 1}{\gamma_M + 1}.$$

This concludes the proof.

To conclude, let us establish Proposition 7.

Proof Let us consider without restriction that $\gamma_m = \gamma_N$. Then, let us observe from the equation (1.47), established in the course of the proof of Proposition 4 that :

$$\theta_N(x,\alpha) = \exp(x)[1-\exp(-x)][\alpha \sum_{i=1}^{N} (\frac{\beta_i}{\gamma_i - 1}) + \frac{1}{2}(1-\exp(-x))] - \sum_{i=1}^{N-1} \theta_i(x,\alpha). \tag{1.59}$$

Let us use this identity to reexpress the algebraic problem (1.36) to be solved as follows :

$$\psi(x,\alpha) = \sum_{i=1}^{N-1} (\gamma_i - \gamma_N)\theta_i(x,\alpha) - T_N(x,\alpha) = 0, \tag{1.60}$$

where we have set :

$$T_N(x,\alpha) = \frac{(\gamma_N + 1)}{2}[1 - \exp(-x)][1 - \frac{\gamma_N - 1}{\gamma_N + 1}(1 + 2\alpha \sum_{i=1}^{N} \frac{\beta_i}{\gamma_i - 1}) \exp(x)]. \tag{1.61}$$

To get a convenient expression for the derivative $x'(\alpha)$, let us differentiate with respect to α the formulation (1.60) when evaluated on $x(\alpha)$:

$$\begin{gathered} \partial_x \psi(x(\alpha),\alpha) \times x'(\alpha) = \\ \frac{\partial T_N}{\partial \alpha}(x(\alpha),\alpha) - \sum_{i=1}^{N-1} (\gamma_i - \gamma_N) \frac{\partial \theta_i}{\partial \alpha}(x(\alpha),\alpha). \end{gathered} \tag{1.62}$$

Easy calculations give the following estimate valid for all $x > 0$:

$$\begin{aligned} \frac{\partial T_N}{\partial \alpha}(x,\alpha) &= \frac{1}{2}(\gamma_N + 1)[1 - \exp(-x)][-2\frac{\gamma_N - 1}{\gamma_N + 1} \sum_{i=1}^{N} \frac{\beta_i}{\gamma_i - 1}] \exp(x) \\ &= -(\gamma_N - 1) \exp(x)[1 - \exp(-x)] \sum_{i=1}^{N} (\frac{\beta_i}{\gamma_i - 1}) < 0. \end{aligned} \tag{1.63}$$

The right hand side of (1.62) thus clearly suggests that $x'(\alpha)$ actually stays negative for sufficiently small $|\gamma_i - \gamma_N|$, since we have already proved $\partial_x \psi(x(\alpha),\alpha) > 0$ for all $\alpha \in (0,1)$. To get this result, let us first recall that for any given set of adiabatic exponents $\{\gamma_i\}_{i=1,\dots,N}$ we have proved the existence of a value $\alpha_c < 1$ such that $x(\alpha)$ strictly decreases for $\alpha \in (\alpha_c, 1]$. Note that α_c depends continuously on the parameters

γ_i entering the definition of the ODE problem (1.28) since does its solution given by the Duhamel formula and hence the mapping $x(\alpha, \{\gamma_i\}_{i=1,...,N})$. The adiabatic exponent γ_N being fixed, there necessarily exists some $\epsilon > 0$ solely depending on γ_N such that $\alpha_c < 1 - \epsilon$ provided that $|\gamma_i - \gamma_N|$ are small enough. This would otherwise rise a contradiction with the property that $\alpha_c = 0$ when the N adiabatic exponents are identical. Consequently, we just have to prove $x'(\alpha) < 0$ for all $\alpha \leq 1 - \epsilon$. Since $x(\alpha)$ decreases in the large, note that $x(\alpha) > x(1-\epsilon) > 0$ so that $\frac{\partial T_N}{\partial \alpha}(x(\alpha), \alpha) > \epsilon_0 > 0$ for all the α under consideration. Here ϵ_0 solely depends on ϵ.
Next, let us use the Duhamel representation formula (1.32) to obtain :

$$\frac{\partial \mathbf{\Theta}}{\partial \alpha}(x(\alpha), \alpha) = \int_0^{x(\alpha)} \exp[\mathcal{M}(x(\alpha) - y)]dy \ \mathbf{a},$$

where $\mathbf{\Theta} = \{\theta_i\}_{i=1,...,N}$. We thus deduce the following crude estimate, valid for all $\alpha \in [0, 1]$:

$$\begin{aligned} \max_{1\leq i\leq N} |\frac{\partial \theta_i}{\partial \alpha}(x(\alpha), \alpha)| &\leq ||\int_0^{x(\alpha)} \exp[\mathcal{M}(x(\alpha) - y)]dy||_\infty \times ||\mathbf{a}||_\infty \\ &\leq \sup_{x\in[0,x(0)]} ||\int_0^x \exp[\mathcal{M}(x-y)]dy||_\infty \times ||\mathbf{a}||_\infty, \end{aligned}$$

with classical notations for the involved vector and matrix norms. Since $x(0)$ is known to be bounded in view of Lemma 3, we thus deduce that $|\sum_{i=1}^{N-1}(\gamma_i - \gamma_N)\frac{\partial \theta_i}{\partial \alpha}(x(\alpha), \alpha)|$ can be made smaller than ϵ_0 when choosing the γ_i sufficiently close to γ_N. This concludes the proof.

1.4 The asymptotic regime and shock solutions

The existence and uniqueness result of Theorem 1 for travelling wave solutions of (1.2) allows us in this section to tackle the asymptotic regime of a vanishing parameter ϵ in (1.1) when defining the notion of shock solutions for the limit system.

1.4.1 Shock solutions

We first deal with shock solutions. Then, rephrazing classical considerations (see Smoller [130] but also LeFloch [90]), starting from a given travelling wave solution $\mathbf{u}(\xi)$ of (1.2) connecting with speed σ a state $\mathbf{u}_-$ to $\mathbf{u}_+$ and a given fixed $\epsilon > 0$, the rescaled function $\mathbf{u}_\epsilon(\xi) = \mathbf{u}(\xi/\epsilon)$ is nothing else a travelling wave solution of (1.1) (*i.e.* with rescaled viscosity coefficents $\epsilon\mu_i$ for $i = 1, ..., N$) connecting the same two states $\mathbf{u}_-$ and $\mathbf{u}_+$ and propagating at the same speed σ. Next and since the proposed rescaling does not affect the (bounded) total variation of $\mathbf{u}$, *i.e.* $TV(\mathbf{u}_\epsilon) = TV(\mathbf{u})$ for all $\epsilon > 0$, the family of travelling wave solutions $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ is easily seen to strongly converge in L^1_{loc} as ϵ goes to zero to the following step function :

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_- & \text{if} \quad x < \sigma t, \\ \mathbf{u}_+ & \text{if} \quad x > \sigma t. \end{cases} \tag{1.64}$$

Following LeFloch [90] (see also Raviart-Sainsaulieu [112]), these considerations thus motivate a natural notion of shock solutions for the system (1.1) in the limit $\epsilon = 0$, referred hereafter as to the limit system.

Definition 1 *The step function (1.64) is said to be a* 1*-shock solution of the limit system if and only if*

$$m = \rho_-(u_- - \sigma) > (\rho c)_-, \tag{1.65}$$

and a 3*-shock solution if and only if*

$$m = \rho_+(u_+ - \sigma) < -(\rho c)_+. \tag{1.66}$$

Let us recall that inequality (1.65) or (1.66) is indeed a necessary (and sufficient) condition for the existence of a travelling wave solution and thus of the limit function under consideration. The following properties for shock solutions are directly inherited from the underlying travelling waves :

Corollary 1 *By construction, a* 1*-shock solution obeys the following Lax shock conditions :*

$$u_+ - c_+ < \sigma < u_- - c_-,$$

and the next N inequalities on the specific entropies :

$$(s_i)_+ \geq (s_i)_- \text{ for } i = 1, ..., N, \tag{1.67}$$

these inequalities being strict for all index i such that $\mu_i > 0$.
Similarly, a 3*-shock solution satisfies :*

$$u_+ + c_+ < \sigma < u_- + c_-,$$

together with for all index $i = 1, ..., N$:

$$(s_i)_+ \leq (s_i)_-, \tag{1.68}$$

with strict inequality as soon as $\mu_i > 0$.
In addition, both types of shock solutions necessarily obey the following Rankine-Hugoniot conditions :

$$-\sigma[\rho] + [\rho u] = 0, \tag{1.69}$$

$$-\sigma[\rho u] + [\rho u^2 + \sum_{i=1}^{N} p_i] = 0, \tag{1.70}$$

together with the additional jump relation :

$$-\sigma[\rho E] + [(\rho E + \sum_{i=1}^{N} p_i)u] = 0, \tag{1.71}$$

which equivalently takes the form of the next Hugoniot-like equation :

$$(\varepsilon_+ - \varepsilon_-) + \frac{1}{2}(p_+ + p_-)(\tau_+ - \tau_-) = 0. \tag{1.72}$$

Proof Smooth solutions of (1.2) are known to obey the additional conservation law (1.5) for governing the total energy, hence also a travelling wave solution. Then integrating the associated ODE form of (1.5) for $\xi \in I\!R$ just gives (1.71). Next, arguments from the classical 3×3 Euler setting exactly apply to infer from equation (1.71) on the ground of (1.69) and (1.70):

$$m\{(\varepsilon_+ - \varepsilon_-) + \frac{1}{2}(p_+ + p_-)(\tau_+ - \tau_-)\} = 0,$$

when understanding p as a (total) pressure and ε as a (total) internal energy. We just get the Hugoniot equation since m cannot be zero.

The reported properties stand as direct extensions of the inequalities met by the shock solutions of the usual 3×3 Euler setting (*i.e.* with $N = 1$). But by deep contrast with this purely conservative setting, a central discrepancy arises from the fact that shock solutions, in the present extended framework, do by definition heavily depend on the choice of the viscosity coefficients μ_i for $i = 1, ..., N$ who gave birth to the underlying travelling wave solutions. Recall that up to this stage, all these coefficients have been tacitly assumed to be fixed but actually arbitrary non negative values can be prescribed under the requirement of a positive sum. The form of the linear reduced ODE problem (1.28)-(1.29) clearly shows that its solution entirely depends on the ratios μ_i/μ for $i = 1, ..., N$ (the adiabatic exponents γ_i being fixed) and so does necessarily the solution of the algebraic equation (1.36). In other words, for a given state $\mathbf{u}_-$ and a velocity σ prescribed under the Lax condition (1.17), the exit state $\mathbf{u}_+$ entering the definition of the step function (1.64) generally differs when considering distinct viscosity ratios. With this respect, shock solutions for the limit system under consideration are sensitive to the small scale properties of a particular diffusive regularization. Such a sensitiveness is actually classical in the setting of hyperbolic systems in non conservation form (see [112], [10], ...) and was thus expected to take place within the present frame. The travelling wave analysis we have proposed actually allows for precisely encoding this small scale sensitiveness when explicitly defining the exact jump in each of the specific entropies in place of the rather vague inequalities (1.67) and (1.68). Indeed, an arbitrary set of non negative viscosities with positive sum being prescribed, then the uniqueness of the exit state $\mathbf{u}_+$ understood as a function of a given left state $\mathbf{u}_-$ and a relevant speed of propagation σ allows us to give the following definition.

Definition 2 *Let be given fixed viscosities $\mu_i \geq 0$ for $i = 1, ..., N$ with $\sum_{i=1}^{N} \mu_i > 0$. Then there exists N associated non negative smooth functions κ_i of $\alpha \in [0, 1]$ and $\beta \in \mathcal{B}_N^+$, such that for the 1-shock solution connecting $\mathbf{u}_-$ to $\mathbf{u}_+$ with speed σ :*

$$s_i(\mathbf{u}_+) - s_i(\mathbf{u}_-) = \kappa_i(\alpha(\mathbf{u}_-, \sigma), \beta(\mathbf{u}_-)), \quad i = 1, ..., N, \tag{1.73}$$

while considering the 3-shock solution which connects $\mathbf{u}_-$ to $\mathbf{u}_+$ with speed σ, one has :

$$s_i(\mathbf{u}_-) - s_i(\mathbf{u}_+) = \kappa_i(\alpha(\mathbf{u}_+, \sigma), \beta(\mathbf{u}_+)), \quad i = 1, ..., N. \tag{1.74}$$

The N functions we have introduced will be referred as to kinetic functions after [15] (see also LeFloch [88]). The N relations in (1.73) obviously serve as a definition formula for each of the kinetic functions, the N identities entering (1.74) being then

a direct consequence of the independence property of the reduced ODE system (1.28) with respect to the sign of the relative Mach number (see Section 1.3).
The following statement summarizes the generalized jump conditions satisfied by a shock solution.

Lemma 4 *Let be given a fixed set of viscosities with positive sum and consider the N associated kinetic functions κ_i. Then a shock solution in the sense of Definition 1 solves the following generalized jump conditions:*

$$\begin{array}{ll}-\sigma(\rho_+ - \rho_-) + ((\rho u)_+ - (\rho u)_-) = 0, & \\ -\sigma((\rho u)_+ - (\rho u)_-) + ((\rho u^2 + p)_+ - (\rho u^2 + p)_-) = 0, & \\ -\sigma((\rho E)_+ - (\rho E)_-) + ((\rho E + p)u_+ - (\rho E + p)u_-) = 0, & \\ -\sigma((\rho s_i)_+ - (\rho s_i)_-) + ((\rho s_i u)_+ - (\rho s_i u)_-) = |\rho_-(u_- - \sigma)|\kappa_i(\alpha,\beta), & i = 1, ..., N-1,\end{array} \tag{1.75}$$

together with:

$$-\sigma((\rho s_N)_+ - (\rho s_N)_-) + ((\rho s_N u)_+ - (\rho s_N u)_-) = |\rho_-(u_- - \sigma)|\kappa_N(\alpha,\beta). \tag{1.76}$$

To conclude notice that such kinetic functions are solely defined when $\alpha \in [0,1]$ but with the property that $\kappa_i(1,\beta) = 0$ for all reduced vector $\beta \in \mathbf{B}$. On the ground of this observation, we choose to extend the domain of definition of these N kinetic functions for all values of $\alpha \in I\!R^+$ when setting by convention for all $\beta \in \mathbf{B}$:

$$\kappa_i(\alpha,\beta) = 0 \text{ for all } \alpha > 1, \quad i = 1, ..., N. \tag{1.77}$$

Such a convention is indeed natural because of the following next result:

Proposition 8 *Let us consider a step function (1.64) with a non zero relative mass flux $m = \rho_-(u_- - \sigma) \in (-(\rho c)_+, (\rho c)_-)$ (i.e. with $\alpha > 1$). Then, this step function cannot solve the generalized jump conditions (1.75)-(1.76) unless $\mathbf{u}_+ = \mathbf{u}_-$.*

In other words, there is no non trivial step functions solutions of (1.75)-(1.76) for a non zero m with $\alpha > 1$. The proof given below will highlight that the validity of (1.77) is indeed necessary to get the conclusion.

Proof By our convention, $\kappa_i(\alpha,\beta) = 0$ for all $i = 1, ..., N$ so that our assumption of a non zero relative mass flux gives necessarily $(s_i)_+ = (s_i)_-$ for all $i = 1, ..., N$. So turning considering the Rankine-Hugoniot condition on the momentum, this one easily recasts as:

$$m^2(\tau_+ - \tau_-) + \sum_{i=1}^{N} s_i(\tau_+^{-\gamma_i} - \tau_-^{-\gamma_i}) = 0.$$

Of course $\tau_+ = \tau_-$ is always solution, but let us assume that this equation can be solved for other values $\tau_+ \neq \tau_-$. Such a non trivial solution must in addition solve the Hugoniot-like equation (1.72) since $m \neq 0$ by assumption (see the proof of Corollary 1). But this Hugoniot-like condition reads:

$$\varepsilon(\tau_+, s_i) - \varepsilon(\tau_-, s_i) + \frac{1}{2}(p(\tau_+, s_i) + p(\tau_-, s_i))(\tau_+ - \tau_-) = 0,$$

where $p(\tau, s_i) = -\partial_\tau \varepsilon(\tau, s_i)$. As a consequence, the above Hugoniot-like condition reads:

$$(\tau_+ - \tau_-)[\frac{1}{2}(p(\tau_+, s_i) + p(\tau_-, s_i)) - 1/\{(\tau_+ - \tau_-)\int_{\tau_-}^{\tau_+} p(v, s_i)dv\}] = 0.$$

But by the strict convexity of the total pressure in the τ variable, the terms inside the brackets is necessarily non zero as soon as $\tau_+ \neq \tau_-$. So this equation cannot be solved for values of τ distinct from τ_-. This completes the proof.

1.4.2 Contact discontinuities

In the classical setting of a single pressure law, shocks are known to coexist with another type of discontinuity: namely the so-called contact discontinuities associated with the linearly degenerate intermediate fields. As expected, such discontinuities arise in the present framework but by contrast to shock solutions, they do not give birth to ambiguities in the limit system. Actually, they are seen below to be already natural weak solutions of parabolic regularization (1.1) of the hyperbolic system under consideration. To that purpose, let us consider the following step function propagating at constant speed u:

$$\mathbf{u}(x,t) = \begin{cases} {}^T(\rho_-, \rho_- u, \{(\rho s_i)_-\}_{i=1,\dots,N}), & x < u\ t, \\ {}^T(\rho_+, \rho_+ u, \{(\rho s_i)_+\}_{i=1,\dots,N}), & x > u\ t, \end{cases} \tag{1.78}$$

under the requirement of a constant total pressure:

$$\sum_{i=1}^{N} p_i(\tau_-, (s_i)_-) = \sum_{i=1}^{N} p_i(\tau_+, (s_i)_+). \tag{1.79}$$

The continuity of both the velocity u and the total pressure of course stems from the property of these two quantities to be Riemann invariants for the intermediate fields of the extracted hyperbolic system (1.4) (see Proposition 1 and the associated discussion). Let us then observe that this step function is a weak solution of (1.1) whatever is the rescaling parameter $\epsilon > 0$. Since the velocity u is constant, one simply has to observe that the following identities are valid in the usual sense of the distributions:

$$\begin{cases} \partial_t \rho + u \partial_x \rho = 0, \\ (\partial_t \rho u + u \partial_x \rho u) + \partial_x \sum_{i=1}^N p_i = 0, \\ \partial_t \rho s_i + u \partial_x \rho s_i = 0, \quad i = 1, \dots, N. \end{cases} \tag{1.80}$$

Indeed and under the requirement of a constant total pressure, the above PDE are nothing but N transport equations at constant speed u. Since the function (1.78)–(1.79) is solution of (1.1) for all $\epsilon > 0$, it obviously remains a weak solution of the limit system. Such a solution will be called a contact discontinuity. Note that since the relative Mach flux $\rho(u - \sigma)$ coming with an arbitrary function step in the form (1.78) is identically zero, the associated reduced number α achieves its value in $\overline{\mathbb{R}}$, namely $\alpha = +\infty$. So that, the convention for extending the domain of definition of the N kinetic functions implies that these all achieve a zero value. Hence, the N last equations in (1.80) can also read under the form of the following trivial identities free from ambiguities:

$$\rho_-(u_- - \sigma)((s_i)_+ - (s_i)_-) = \rho_-(u_- - \sigma)\kappa_i(\infty, \beta), \quad i = 1, ..., N. \tag{1.81}$$

These identities just express that the jump in each of the specific entropies s_i is arbitrary. Notice in addition that contact discontinuities trivially also satisfy in the sense of the distributions the following conservation law for the total energy

$$\partial_t \rho E + \partial_x(\rho E + \sum_{i=1}^{N} p_i)u = 0, \tag{1.82}$$

since the relative mass flux $m = 0$ (see the proof of Corollary 1). Summarizing the above observation, we state:

Lemma 5 *Let be given a fixed set of viscosities with positive sum and consider the N associated kinetic functions κ_i. Then a contact discontinuity solution solves the following jump conditions:*

$$\begin{aligned}
&-\sigma(\rho_+ - \rho_-) + ((\rho u)_+ - (\rho u)_-) = 0,\\
&-\sigma((\rho u)_+ - (\rho u)_-) + ((\rho u^2 + p)_+ - (\rho u^2 + p)_-) = 0,\\
&-\sigma((\rho E)_+ - (\rho E)_-) + ((\rho E + p)u_+ - (\rho E + p)u_-) = 0,\\
&-\sigma((\rho s_i)_+ - (\rho s_i)_-) + ((\rho s_i u)_+ - (\rho s_i u)_-) = |\rho_-(u_- - \sigma)|\kappa_i(\alpha, \beta) = 0, \quad i = 1, ..., N-1,
\end{aligned} \tag{1.83}$$

together with:

$$-\sigma((\rho s_N)_+ - (\rho s_N)_-) + ((\rho s_N u)_+ - (\rho s_N u)_-) = |\rho_-(u_- - \sigma)|\kappa_N(\alpha, \beta) = 0. \tag{1.84}$$

1.4.3 The limit first order system

In this subsection, we focus ourselves on piecewise Lipschitz-continuous limit functions $\mathbf{u}$ obtained as limits of families of solutions $\{\mathbf{u}\}_{\epsilon>0}$ of (1.1) as ϵ goes to zero. On the ground of the N kinetic functions derived in subsection 1.4.1, we introduce with little abuse in the notations N bounded non negative Borel measures $\kappa_i(\mathbf{u})$ defined from any given Lipschitz-continuous function $\mathbf{u}$ as follows. In the zone of smoothness of the function $\mathbf{u}$, these measures naturally identically vanishe while across a discontinuity in $\mathbf{u}$ separating the states $\mathbf{u}_-$ and $\mathbf{u}_+$ and propagating at speed σ:

$$\kappa_i(\mathbf{u}) = \kappa_i(\alpha, \beta)\delta_{x-\sigma t}, \quad i = 1, ..., N, \tag{1.85}$$

where depending on the sign of the relative mass flux $m = \rho_-(u_- - \sigma)$, the dimensionless quantities α and $\beta = \{\beta_i\}_{i=1,...,N}$ are built from $\mathbf{u}_-$ ($m \geq 0$) or $\mathbf{u}_+$ ($m < 0$). Let us recall that by convention $\kappa_i(\alpha, \beta) = 0$ as soon as $\alpha \in [1, +\infty)$: *i.e.* when $m \in [-(\rho c)_+, (\rho c)_-]$. Consequently and when addressing contact discontinuities, *i.e.* with $m = 0$, the measures under consideration are identically zero. These measures are therefore only non trivial on shock discontinuities, and by construction they stay bounded and positive for all parameter $\alpha \in [0, 1]$. Equipped with these bounded measures, we then propose after Berthon et al. [15]:

Definition 3 *A piecewise Lipschitz-continuous funtion is said to be a solution of (1.1) in the asymptotic regime $\epsilon \to 0$, if and only if it obeys in the sense of the distributions the following first order system with measures source terms:*

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i) = 0, \\ \partial_t \rho E + \partial_x (\rho E + \sum_{i=1}^N p_i) u = 0 \\ \partial_t \rho s_i + \partial_x \rho s_i u = \kappa_i(\mathbf{u}), \quad i = 1, ..., N-1, \end{cases} \tag{1.86}$$

together with the additional law

$$\partial_t \rho s_N + \partial_x \rho s_N u = \kappa_N(\mathbf{u}). \tag{1.87}$$

The relevance of the proposed definition simply stems from the Lemmae 4 and 5.

1.5 The Riemann problem

This section aims at proving that the Riemann problem for the first order limit system (1.86)-(1.87) admits an unique solution for arbitrary states $\mathbf{u}_L$ and $\mathbf{u}_R$ in Ω, provided that these two states do not yield vacuum. Occurence of vacuum will be given a precise meaning hereafter, which will be seen to naturally extend the no-vacuum condition met in the setting of a single pressure law. To shade some light on the forthcoming developments, let us first comment the eigenstructure of the limit system (1.86). As it is well-known, this eigenstructure has to be considered within the frame of smooth solutions, that is to stay when all the measures identically vanishes. With this respect, the resulting system in conservation form is just equivalent to the first order underlying system (1.4) we have studied in the second section. The system (1.86) then shares exactly the same hyperbolicity properties stated in Proposition 1. Classical considerations from the respective properties of the fields under consideration indicate that the Riemann solution is at most made of four distinct states, namely $\mathbf{u}_L$, $\mathbf{u}_L^\star$, $\mathbf{u}_R^\star$ and $\mathbf{u}_R$, separated by three simple waves. The property of genuine non linearity of the two extreme fields is responsible for the property that the extreme waves are indeed simple, while the intermediate fields associated with the same eigenvalue can only result in a contact discontinuity. This waves pattern thus coincides with the one met in the case of a single pressure law. We shall thus adopt similar steps in the analysis of the Riemann solution when deriving at first a notion of shock curves and of rarefaction curves for the two extreme fields. Then, the property of a constant total pressure $p = \sum_{i=1}^N p_i$ and constant velocity u valid for the intermediate contact discontinuity will naturally suggest us to determine the projection of these two families of curves in the (p, u) plane. The respective monotonicity properties and the asymptotic behaviours of these projections will ensure existence and uniqueness of an intersection point: namely the total pressure and velocity at the expected contact discontinuity.

1.5.1 Shock curves

For any given fixed state $\mathbf{u}_L \in \Omega$, let us first define the 1-shock curve attached to the left state $\mathbf{u}_L$,

$$\mathcal{S}_1(\mathbf{u}_L) = \{\mathbf{u} \in \Omega, \mathbf{u}_L \ \underline{\text{1-shock}} \ \mathbf{u}\},$$

as the subset of Ω made of the totality of states $\mathbf{u}$ which can be connected on the right to $\mathbf{u}_L$ by an 1-shock solution in the sense of Definition 1.
Similarly and for a given state $\mathbf{u}_R \in \Omega$, let us then introduce the 3-shock curve attached to the right state $\mathbf{u}_R$,

$$\mathcal{S}_3(\mathbf{u}_R) = \{\mathbf{u} \in \Omega, \mathbf{u} \ \underline{\text{3-shock}} \ \mathbf{u}_R\},$$

made by definition of the states $\mathbf{u}$ which can be connected on the left to $\mathbf{u}_R$ by an 3-shock solution.
Motivated by the general considerations given at the beginning of this section, we propose to determine the projection of these two subsets on the (p, u) plane. We shall give a complete derivation when restricting ourselves to $\mathcal{S}_1(\mathbf{u}_L)$, the required characterization of $\mathcal{S}_3(\mathbf{u}_R)$ will follow along the same steps.
By the derivation of the family of 1-shock solutions with $\mathbf{u}_L$ as a left state, α defined in (1.30) serves as a natural parameter along $\mathcal{S}_1(\mathbf{u}_L)$:

$$\mathcal{S}_1(\mathbf{u}_L) = \{\mathbf{u}(\alpha) = (\tau(\alpha), u(\alpha), p(\alpha), \{s_i(\alpha)\}_{i=1,\ldots,N-1}) \text{ with } \alpha \in [0,1]\},$$

but the following easy statement proves that the total pressure p stands for another admissible parameter, actually more relevant to our purpose :

Lemma 6 *The total pressure is a smooth strictly decreasing function of α in $[0,1]$ which maps $[0,1]$ onto $[p_L, +\infty]$.*
Proof Using the notations of the previous section, the total pressure obeys by construction along $\mathcal{S}_1(\mathbf{u}_L)$:

$$p(\alpha) = p_L + \frac{\rho_L c_L^2}{\alpha}(1 - \frac{\tau(\alpha)}{\tau_L}),$$

where the function $\tau(\alpha) = \tau_L \exp(-x(\alpha))$ has been shown to smoothly increase in the large for α in $[0,1]$ from some τ_{min} in $(0, \tau_L)$ to τ_L while being necessarily strictly increasing for α close to 1 (*i.e.* when $\tau(\alpha)$ is close to τ_L). The required conclusion then easily follows.

Equipped with this result, we then get the required characterization of the projection of $\mathcal{S}_1(\mathbf{u}_L)$ onto the (p, u) plane.

Proposition 9 *Let be given $\mathbf{u}_L$ in Ω. States $\mathbf{u}$ along $\mathcal{S}_1(\mathbf{u}_L)$ are given as a smooth function*

$$\mathbf{u}_1(p) = \{\tau_1(p), u_1(p), p, \{(s_i)_1(p)\}_{i=1,\ldots,N-1}\}$$

of p in $[p_L, +\infty)$. Moreover, the velocity u_1 is a smooth strictly decreasing function which maps $[p_L, +\infty)$ onto $(-\infty, u_L]$.

Proof Let us first observe that the function τ is a smooth decreasing (in the large) function of p in $[p_L, +\infty)$ with range $(\tau_{min}, \tau_L]$ for some τ_{min} in $(0, \tau_L)$, while being necessarily strictly decreasing for p close to p_L.
Let us then prove that for all $p \geq p_L$, the velocity reads :

$$u_1(p) = u_L - ((\tau_L - \tau_1(p))(p - p_L))^{1/2}. \tag{1.88}$$

Indeed, the two identities

$$u_1(p) - u_L = m(\tau_1(p) - \tau_L), \quad p - p_L = m^2(\tau_L - \tau_1(p)),$$

readily implies that :

$$(u_1(p) - u_L)^2 = (p - p_L)(\tau_L - \tau_1(p)) \geq 0, \quad p \geq p_L.$$

But by the Lax condition (1.17) for a 1-shock solution, $m > (\rho c)_L$ while $\tau_1(p) \leq \tau_L$ for all $p \geq p_L$, we thus necessarily have $u_1(p) \leq u_L$ and hence the representation formula (1.88). The conclusion easily follows from the properties of the function τ_1 we have just reported.

To conclude this paragraph, let us state concerning the projection of the shock curve $\mathcal{S}_3(\mathbf{u}_R)$ onto the (p, u) plane :

Proposition 10 *For any given fixed $\mathbf{u}_R$ in Ω, states $\mathbf{u}$ along $\mathcal{S}_3(\mathbf{u}_R)$ are given as a smooth function*

$$\mathbf{u}_3(p) = \{\tau_3(p), u_3(p), p, \{(s_i)_3(p)\}_{i=1,\ldots,N-1}\}$$

of p in $[p_R, +\infty)$, where the function u_3 strictly increases with p, and with range $[u_R, +\infty)$.

Proof Identical steps are in order except that the Lax condition (1.17) for a 3-shock solution says $m < -(\rho c)_R$ and now imply the following representation formula :

$$u_3(p) = u_R + ((\tau_R - \tau_3(p))(p - p_R))^{1/2}, \quad p \geq p_R.$$

The conclusion then follows.

1.5.2 Rarefaction waves and rarefaction curves

By definition, a rarefaction wave is a smooth solution of the asymptotic system (1.2) which is self-similar in the variable $\xi = x/t$. The smoothness assumption thus makes a rarefaction wave to be a particular solution, say $\mathbf{w}(\xi)$, of the system (1.86) with identically vanishing measures :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i(\rho, s_i)) = 0, \\ \partial_t \rho s_i + \partial_x \rho s_i u = 0, \quad i = 1, ..., N. \end{cases} \tag{1.89}$$

Standard arguments from hyperbolic systems in full conservation form (see for instance [56], [130]) thus immediately apply to give that the self-similar solution under consideration must solve an $(N+2)$ ODE system in the form :

$$d_\xi \mathbf{w}(\xi) = r_i(\mathbf{w}(\xi)), \tag{1.90}$$

for some right eigenvector $r_i(\mathbf{u})$ of (1.89), while satisfying for the same index i along the orbit of (1.90) :

$$\lambda_i(\mathbf{w}(\xi)) = \xi. \tag{1.91}$$

Smoothness of the solution of (1.90) is known to imply the non linearity of the field under consideration (since in view of (1.91), we must have $d_\xi \lambda_i(\mathbf{w}(\xi)) = \nabla\lambda_i.r_i(\mathbf{w}(\xi)) = 1$). As a consequence, only the two extreme fields, namely either $i=1$ or $i=3$ (actually genuinely non linear) can give rise to rarefaction wave solution.
The following result states that each of the two $(N+2)$ ODE system (1.90) of interest, *i.e.* with $i=1$ or $i=3$, admits $(N+1)$ algebraic invariants with linearly independent gradients (*i.e.* a full set of Riemann invariants, see again [56] or [130]).These are straightforward extensions of the two invariants known in the setting of a single pressure law and their existence allows for defining in the present framework, the notion of (two) families of rarefaction curves.

Lemma 7 *For some given state $\mathbf{u}_L \in \Omega$, let us define the* 1*-rarefaction curve attached to the left state $\mathbf{u}_L$ as the following subset of Ω:*

$$\mathcal{R}_1(\mathbf{u}_L) = \{\mathbf{u} \in \Omega, \mathbf{u}_L \ \underline{\textit{1-rarefaction}}\ \mathbf{u}\},$$

namely the set of right states $\mathbf{u}$ which can be connected to $\mathbf{u}_L$ by an admissible 1-rarefaction wave. Then $\mathbf{u}$ belongs to $\mathcal{R}_1(\mathbf{u}_L)$ if and only if:

$$u - c(\rho, \{s_i\}_{i=1,...,N}) > u_L - c_L, \quad \textit{as soon as } \mathbf{u} \neq \mathbf{u}_L, \tag{1.92}$$

and

$$s_i = (s_i)_L, \quad i = 1, ..., N, \tag{1.93}$$

$$u + g_L(\rho, \{(s_i)_L\}_{i=1,...,N}) = u_L + g_L(\rho_L, \{(s_i)_L\}_{i=1,...,N}), \tag{1.94}$$

where by definition:

$$g_L(\rho, \{s_i\}_{i=1,...,N}) = \int_{\rho_L}^{\rho} c(v, \{s_i\}_{i=1,...,N}) \frac{dv}{v},$$

with in particular $g_L(\rho_L, \{(s_i)_L\}_{i=1,...,N}) = 0$.
Symmetrically, define the 3*-rarefaction curve attached to a given right state $\mathbf{u}_R \in \Omega$ by:*

$$\mathcal{R}_3(\mathbf{u}_R) = \{\mathbf{u} \in \Omega, \mathbf{u} \ \underline{\textit{3-rarefaction}}\ \mathbf{u}_R\},$$

namely the set of left states $\mathbf{u}$ which can be connected to $\mathbf{u}_R$ by an admissible 3-rarefaction wave. Then $\mathbf{u} \in \mathcal{R}_3(\mathbf{u}_R)$ if and only if:

$$u + c(\rho, \{s_i\}_{i=1,...,N}) < u_R + c_R, \quad \textit{as soon as } \mathbf{u} \neq \mathbf{u}_R,$$

and

$$s_i = (s_i)_R, \quad i = 1, ..., N,$$

$$u - g_R(\rho, \{(s_i)_R\}_{i=1,...,N}) = u_R - g_R(\rho_R, \{(s_i)_R\}_{i=1,...,N}),$$

where by definition:

$$g_R(\rho, \{s_i\}_{i=1,...,N}) = \int_{\rho_R}^{\rho} c(v, \{s_i\}_{i=1,...,N}) \frac{dv}{v}.$$

Proof We shall only be concerned here with the characterization of the set of states $\mathcal{R}_1(\mathbf{u}_L)$. Invoking the special form of the smooth solution under consideration, the N advection equations governing each of the specific entropies (away from vacuum):

$$\partial_t s_i + u\partial_x s_i = 0, \quad i = 1, ..., N,$$

just recast as:

$$(u(\xi) - \xi)d_\xi s_i(\xi) = 0, \quad i = 1, ..., N.$$

But since the identity (1.91) cannot hold for the intermediate (linearly degenerate) field, necessarily $u(\xi) \neq \xi$ and henceforth $d_\xi s_i(\xi) = 0$ for all $i = 1, ..., N$. This is nothing but the N Riemann invariants listed in (1.93).
Consequently, the total pressure p is solely a function of the density ρ for a rarefaction wave solution and the last Riemann invariant (1.94) just follows from the well-known theory of the isentropic Euler equations (see [56]).
To conclude the proof, let us recall that the condition (1.92) is immediately inferred from the genuine non linearity of the field under consideration (*i.e.* $d_\xi\lambda_1(\mathbf{w}(\xi)) = 1$) under the requirement that $\xi(\mathbf{u}) > \xi(\mathbf{u}_L)$: indeed $\mathbf{u}$ on the right is by definition connected to $\mathbf{u}_L$ on the left.

Equipped with this result, we are now in position to determine the projections of these two families of rarefaction curves onto the total pressure-velocity plane. This is precisely the matter of the next statement.

Proposition 11 *For any given state $\mathbf{u}_L \in \Omega$, the total pressure p serves as a parameter along $\mathcal{R}_1(\mathbf{u}_L)$ when defining each state $\mathbf{u} \in \mathcal{R}_1(\mathbf{u}_L)$ from a smooth function $\mathbf{u}_1(p) = (\tau_1(p), u_1(p), p, \{s_i\}_{i=1,...,N-1})$ of p in $(0, p_L]$ where the velocity u_1 is a smooth strictly decreasing function, mapping $(0, p_L]$ onto $[u_L, u_{max}(\mathbf{u}_L))$ for some $u_{max}(\mathbf{u}_L)$ in $(u_L, +\infty)$.*
Similarly and for any given $\mathbf{u}_R \in \Omega$, states $\mathbf{u}$ along $\mathcal{R}_3(\mathbf{u}_R)$ are given as a smooth function $\mathbf{u}_3(p) = (\tau_3(p), u_3(p), p, \{s_i\}_{i=1,...,N-1})$ of p in $(0, p_R]$, u_3 being a smooth strictly increasing function which maps $(0, p_R]$ onto $(u_{min}(\mathbf{u}_R), u_R]$ for some $u_{min}(\mathbf{u}_R)$ in $(-\infty, u_R)$.

Proof The proof we propose only addresses the 1-rarefaction curve $\mathcal{R}_1(\mathbf{u}_L)$. As far as $\mathcal{R}_3(\mathbf{u}_R)$ is concerned, the result will follow from simple adaptations of the arguments we now develop. Since the N specific entropies s_i are Riemann invariants, we first observe that the density ρ serves as a natural parameter along $\mathcal{R}_1(\mathbf{u}_L)$ which domain is actually restricted by the condition (with little abuse in the notations):

$$\{u - c\}(\rho) \geq \{u - c\}(\rho_L). \tag{1.95}$$

Indeed, observing that

$$\frac{d}{d\rho}\{u - c\}(\rho) = -\frac{1}{2\rho c}\sum_{i=1}^{N} \gamma_i(\gamma_i + 1)(s_i)_L \rho^{\gamma_i - 1} < 0,$$

we infer that ρ must be kept in $(0, \rho_L]$ for (1.95) to hold true. Thus, the Riemann invariant (1.94) defines the velocity u as a smooth strictly decreasing function of the density which mapps $(0, \rho_L]$ onto $[u_L, u_{max})$ where by definition, u_{max} denotes a real number in $[u_L, +\infty[$ given by :

$$u_{max} = u_L + \lim_{\rho \to 0^+} \int_{\rho}^{\rho_L} c(v, \{(s_i)_L\}_{i=1,...,N}) \frac{dv}{v}. \tag{1.96}$$

The boundedness of u_{max} clearly holds by invoking the specific assumptions on the adiabatic exponents $\gamma_i > 1$, $i = 1, ..., N$.
Next and again because the N specific entropies are Riemann invariants, the total pressure

$$p(\rho) = \sum_{i=1}^{N} (s_i)_L \rho^{\gamma_i}$$

is a smoothly strictly increasing function of ρ in $(0, \rho_L]$ and with range $(0, p_L]$ (since $\gamma_i > 1$ for all $i = 1, ..., N$). It thus follows that along $\mathcal{R}_1(\mathbf{u}_L)$, the covolume τ is a smooth decreasing function $\tau_1(p)$ of $p \in (0, p_L]$ while the velocity u is a smooth decreasing function $u_1(p)$ mapping $(0, p_L]$ onto $[u_L, u_{max})$ with u_{max} given by (1.96). This concludes the proof.

1.5.3 Existence and uniqueness

Considering an arbitrary fixed initial data $(\mathbf{u}_L, \mathbf{u}_R)$ throughout this section, we are now in a position to define the following two curves of interest :

$$\mathcal{C}_1(\mathbf{u}_L) = \mathcal{S}_1(\mathbf{u}_L) \cup \mathcal{R}_1(\mathbf{u}_L), \quad \text{and} \quad \mathcal{C}_3(\mathbf{u}_R) = \mathcal{S}_3(\mathbf{u}_R) \cup \mathcal{R}_3(\mathbf{u}_R).$$

By construction,

$$\mathcal{C}_1(\mathbf{u}_L) = \{\mathbf{u} \in \Omega, \mathbf{u}_L \ \underline{\text{1-wave}} \ \mathbf{u}\},$$

is nothing but the set of states $\mathbf{u}$ which can be connected on the right to the left state $\mathbf{u}_L$ by a simple 1-wave (either an admissible shock or an admissible rarefaction). Then,

$$\mathcal{C}_3(\mathbf{u}_R) = \{\mathbf{u} \in \Omega, \mathbf{u} \ \underline{\text{3-wave}} \ \mathbf{u}_R\},$$

consists in states $\mathbf{u}$ which can be connected on the left to the right state $\mathbf{u}_R$ by an admissible simple 3-wave.
The projection of these two curves onto the (p, u) plane then have the following distinctive properties :

Proposition 12 *Along the curve $\mathcal{C}_1(\mathbf{u}_L)$, the velocity u_1 is a smooth strictly decreasing function of the total pressure p mapping $(0, +\infty)$ onto $(-\infty, u_{max}(\mathbf{u}_L))$ for some $u_{max}(\mathbf{u}_L)$ in $(u_L, +\infty)$ while along the curve $\mathcal{C}_3(\mathbf{u}_R)$, the velocity u_3 smoothly strictly increases with p in $(0, +\infty)$ and with range $(u_{min}(\mathbf{u}_R), +\infty)$ for some $u_{min}(\mathbf{u}_R)$ in $(-\infty, u_R)$.*

Proof We just have to check that the corresponding shock curves and rarefaction curves achieve a smooth connection. Focusing ourselves on $\mathcal{C}_1(\mathbf{u}_L)$, let us prove that $\mathcal{S}_1(\mathbf{u}_L)$ and $\mathcal{R}_1(\mathbf{u}_L)$ are actually tangent at $p = p_L$. Indeed, invoking the smooth representation formula (1.94) valid along $\mathcal{R}_1(\mathbf{u}_L)$, we first observe that the velocity u_1 when understood as a (smooth) function of the density admits a left derivative at $\rho = \rho_L$ explicitly given by $-c(\rho_L, \{(s_i)_L\}_{i=1,...,N})/\rho_L$ while the form of the total pressure

$p(\rho) = \sum_{i=1}^{N} \rho^{\gamma_i}(s_i)_L$ along $\mathcal{R}_1(\mathbf{u}_L)$ allows to compute $p'(\rho_L) = c^2(\rho_L, \{(s_i)_L\}_{i=1,\dots,N})$. Consequently, the velocity function $u_1(p)$ admits a left derivative at $p = p_L$ given by $-1/(\rho c)_L$.

Next turning considering the companion representation formula (1.88) along $\mathcal{S}_1(\mathbf{u}_L)$, we have in one hand and for all $p > p_L$:

$$\frac{u_1(p) - u_1(p_L)}{p - p_L} = -(\frac{\tau_1(p_L) - \tau_1(p)}{p - p_L})^{1/2}, \tag{1.97}$$

while on the second hand and again for all $p > p_L$:

$$\frac{\tau_1(p) - \tau_1(p_L)}{p - p_L} = -\frac{1}{m^2(p)} = -\frac{\alpha(p)}{(\rho c)_L^2}. \tag{1.98}$$

Since the covolume is a $\mathcal{C}^1$ function of the total pressure for $p \in [p_L, +\infty)$, (1.97) and (1.98) imply that the velocity function $u_1(p)$ admits a right derivative at $p = p_L$ given by $-1/(\rho c)_L$ (since $\alpha(p_L) = 1$). This is nothing but the required result. Similar arguments apply to prove the smooth connexion of $\mathcal{S}_3(\mathbf{u}_R)$ and $\mathcal{R}_3(\mathbf{u}_R)$.

As a direct consequence of the reported distinct monotonicity properties of the velocity functions $u_1(p)$ and $u_3(p)$ and their asymptotic behaviours, the respective projections of the curves $\mathcal{C}_1(\mathbf{u}_L)$ and $\mathcal{C}_3(\mathbf{u}_R)$ must necessarily intersect at a given pair $(u_\star, p_\star)$ as soon as

$$u_{min}(\mathbf{u}_R) < u_{max}(\mathbf{u}_L).$$

Therefore, the Riemann problem admits at least one solution given with clear notations by :

$$\mathbf{u}_L \ \underline{\text{1-wave}}\ \mathbf{u}_1(p_\star)\ \underline{\text{contact}}\ \mathbf{u}_3(p_\star)\ \underline{\text{3-wave}}\ \mathbf{u}_R.$$

with

$$\mathbf{u}_1(p_\star) = (\tau_1(p_\star), u_\star, p_\star, \{(s_i)_1(p_\star)\}_{i=1,\dots,N}),$$

and

$$\mathbf{u}_3(p_\star) = (\tau_3(p_\star), u_\star, p_\star, \{(s_i)_3(p_\star)\}_{i=1,\dots,N}).$$

But the strict monotonicity properties of the velocity functions under consideration imply that the pair $(u_\star, p_\star)$ is necessarily unique as soon as it exists.

Next turning considering the case $u_{max}(\mathbf{u}_L) \leq u_{min}(\mathbf{u}_R)$, this instance rises the same difficulty as in the setting of the usual Euler equations (*i.e.* with $N = 1$), namely the occurence of vacuum. In such an issue, it is known that the Riemann problem has no solution unless one specifically allows for a "void" state. We shall not address this particular construction and we refer the interested reader to Smith [129] and the references therein.

We have therefore proved :

Theorem 2 *The Riemann problem with initial data* $(\mathbf{u}_L, \mathbf{u}_R)$ *admits an unique solution away from vacuum, i.e. if*

$$u_{min}(\mathbf{u}_R) < u_{max}(\mathbf{u}_L),$$

where these two particular velocities have been defined in Proposition 11.