Chapitre 2

Numerical approximation of the Navier-Stokes equations with several independent specific entropies

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Numerical approximation of the Navier-Stokes equations with several independent specific entropies

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Résumé

The present work is devoted to the numerical approximation of the solutions of convective-diffusive systems which can be understood as natural extensions of the classical Navier-Stokes (NS) equations. Solutions of these systems are governed by N ($N \geq 2$) independent pressure laws or equivalently N independent specific entropies. Several models from the physics actually enter the present framework. These extended forms of the usual NS equations (i.e. when $N = 1$) are seen to share many properties but with the very difference that, generally speaking, they cannot be recast in full conservation form. This property is known to make the definition of the endpoints of the Travelling Wave (TW) solutions sensitive with respect to the shape of the diffusive tensor. Motivated by the work of Berthon [10] where $N = 2$, here we exhibit $(N-1)$ Generalized Rankine-Hugoniot (GRH) conditions that reflect this sensitivity. We underline the reason why classical algorithms can only fail in satisfying these GRH relations and we show how to enforce them for validity at the discrete level when extending [10] to $N \geq 2$ arbitrary general pressure laws. Several numerical evidences are proposed.

2.1 The mathematical model and some basic properties

In this work, we are interested in the numerical approximation of the solutions of the following convective-diffusive system (in one space dimension for simplicity) :

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \rho u = 0, \qquad x \in I\!R, \quad t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \sum_{i=1}^N p_i) = \partial_x ((\sum_{i=1}^N \mu_i) \partial_x u), \\ \partial_t \rho \epsilon_i + \partial_x (\rho \epsilon_i u) + p_i \partial_x u = \mu_i (\partial_x u)^2, \quad i = 1, ..., N, \end{array} \right. \tag{2.1}$$

where with classical notations, $\rho, \rho u$ and $\{\rho\epsilon_i\}_{i=1,...,N}$ respectively denote the density, the momentum and N independent internal energies. Before addressing the required closure relations, let us stress that (2.1) is nothing but a natural extension of the usual NS equations : namely when considering a single pressure law. Here, N ($N \geq 2$) independent pressure laws $p_i(\rho, \rho\epsilon_i)$ are involved via N independent governing equations for each of the $\rho\epsilon_i$. In (2.1) and for $i = 1, \ldots, N$, μ_i ($\mu_i \geq 0$) denotes a smooth mapping standing for the viscosity law associated with the internal energy $\rho\epsilon_i$. It will be always assumed that $\mu := \sum_{i=1}^N \mu_i > 0$. Depending on the choice of the closure equations for defining each of the μ_i, several models from the physics enter the present framework. Let us quote for instance compressible turbulence models like the $k - \epsilon$ model (see [13]), and more generally multi-scale models (see [19]) for which

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a laminar pressure and some turbulent pressures are involved, or models coming from the plasma physics (see [34], [41]).
Now considering the required closure equations for defining each of the N pressure laws, we shall assume that according to the second principle of the thermodynamics:

$$-T_i(\tau,\epsilon_i)ds_i = d\epsilon_i + p_i(\tau,\epsilon_i)d\tau, \quad \tau = 1/\rho, \tag{2.2}$$

defines a strictly convex smooth mapping $(\tau,\epsilon_i) \to s_i(\tau,\epsilon_i) > 0$ for any given $i \in \{1,..,N\}$ where the temperature T_i is classically assumed to stay positive. As a well known consequence, the well defined mapping $(\tau,s_i) \to \varepsilon_i(\tau,s_i)$ is then strictly convex, as well as the mapping $(\rho,\rho\varepsilon_i) \to \{\rho s_i\}(\rho,\rho\varepsilon_i) := \rho s_i(1/\rho,\rho\varepsilon_i/\rho)$ and $(\rho,\rho s_i) \to \{\rho\varepsilon_i\}(\rho,\rho s_i) := \rho\varepsilon_i(1/\rho,\rho s_i/\rho)$ with some little abuse in the notations. Each energy law is assumed to obey in addition the well-known Weyl's assumptions (see [57] for the details). We shall use in particular the following asymptotic conditions:

$$\lim_{\rho s_i\to 0^+}\rho\varepsilon_i(\rho,\rho s_i) = +\infty \text{ and } \lim_{\rho s_i\to +\infty}\rho\varepsilon_i(\rho,\rho s_i) = 0 \ (i.e. \lim_{\rho\varepsilon_i\to 0^+}\rho s_i(\rho,\rho\varepsilon_i) = +\infty).$$

Notice that all the above assumptions are quite classical within the frame of the usual NS equations (i.e. with $N = 1$). Owing to these assumptions, our first statement highlights the relationships with the usual setting, when defining the following natural phase space for (2.1):

$$\Omega = \{\mathbf{v} := (\rho,\rho u,\{\rho\epsilon_i\}_{1\le i\le N}) \in I\!R^{N+2}/\rho > 0,\ \rho u \in I\!R,\ \rho\epsilon_i > 0,\ 1 \le i \le N\}. \tag{2.3}$$

Lemma 8
The first order extracted system from (2.1) is hyperbolic over Ω, with the following increasingly ordered eigenvalues:

$$\lambda_1(\mathbf{v}) = u - c \le \lambda_2(\mathbf{v}) = \ldots = \lambda_{N+1}(\mathbf{v}) = u, \le \lambda_{N+2}(\mathbf{v}) = u + c,\ c^2(\mathbf{v}) = \sum_{i=1}^N c_i^2(\mathbf{v}),$$

where each partial sound speed is such that $c_i^2(\mathbf{v}) := (\partial_\rho p_i)_{s_i} > 0$. The $1-$ and $(N+2)-$ fields are genuinely nonlinear, all the other fields being linearly degenerate.
The proof of this easy result is omitted.
The system (2.1) naturally writes in non conservation form. This rises the question of the existence of an admissible change of variables that recasts (2.1) in full conservation form. With this respect, the next claim gives all the additional non trivial equations for the smooth solutions of (2.1) (see [19] or [10] when $N = 2$):

Theorem 3
Smooth solutions of (2.1) obey the following conservation law:

$$\partial_t\rho E + \partial_x(\rho E + \sum_{i=1}^N p_i)u = \partial_x((\sum_{i=1}^N \mu_i)u\partial_x u), \tag{2.4}$$

where the total energy reads $\rho E = (\rho u)^2/2\rho + \sum_{i=1}^N \rho\epsilon_i$. In addition, such solutions satisfy:

$$\partial_t\rho s_i + \partial_x\rho s_i u = -\frac{\mu_i}{T_i}(\partial_x u)^2, \quad i = 1,...,N. \tag{2.5}$$

Up to classical nonlinear transformations in the $\{s_i\}_{1\le i\le N}$, these are the only additional non trivial equations (without special assumptions on the $\{\mu_i\}_{1\le i\le N}$).

As a consequence, the system (2.1) cannot be recast generally speaking in full conservation form. However, let us rephrase the above result when investigating suitable change of variables for (2.1). Obviously, the three variables ρ, ρu and ρE are natural candidates since they are governed by conservation laws. Next and for completeness, let us consider without restriction the set $\{\rho s_i\}_{1\leq i\leq N-1}$ to define $\mathbf{u} := (\rho, \rho u, \rho E, \{\rho s_i\}_{1\leq i\leq N-1})$ as an admissible unknown. Then, smooth solutions of (2.1) equivalently satisfy :

$$\begin{cases} \partial_t\rho + \partial_x\rho u = 0, \qquad x \in I\!R, \quad t > 0, \\ \partial_t\rho u + \partial_x(\rho u^2 + \sum_{i=1}^N p_i) = \partial_x((\sum_{i=1}^N \mu_i)\partial_x u), \\ \partial_t\rho E + \partial_x(\rho E + \sum_{i=1}^N p_i)u = \partial_x((\sum_{i=1}^N \mu_i)u\partial_x u), \\ \partial_t\rho s_i + \partial_x\rho s_i u = -\frac{\mu_i}{T_i}(\partial_x u)^2, \quad i = 1, ..., N-1. \end{cases} \tag{2.6}$$

Let us notice that the first order underlying system in (2.6) is fully conservative. For conciseness in the forthcoming developments, we right away introduce the following condensed notation for (2.6):

$$\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u}) = \mathcal{R}(\mathbf{u}, \partial_x(\mathcal{B}(\mathbf{u})\partial_x\mathbf{u})).$$

The definition of the tensor $\mathcal{B}$ follows when writing $(\partial_x u)^2 = \partial_x(u\partial_x u) - u\partial_{xx}u$. Now, let us emphasize that whitout further condition, smooth solutions of (2.6) automatically obey with equality the additional PDE :

$$\partial_t\rho s_N(\mathbf{u}) + \partial_x\rho s_N(\mathbf{u})u = -\frac{\mu_N}{T_N}(\partial_x u)^2. \tag{2.7}$$

Notice that as a consequence of the convexity properties we have put forward at the beginning of this section, the mapping $\mathbf{u} \to \rho s_N(\mathbf{u})$ is strictly convex. Hence, the pair $(\rho s_N, \rho s_N u)$ is nothing but a Lax entropy pair but with a prescribed rate of entropy dissipation. To assess this issue, let us observe after Berthon [10] that despite the identities (2.5) might be understood as N fully independent equations, these do actually evolve proportionally according to :

Corollary 2
Smooth solutions of (2.6) obey :

$$\mu_N T_i\{\partial_t\rho s_i + \partial_x\rho s_i u\} = \mu_i T_N\{\partial_t\rho s_N(\mathbf{u}) + \partial_x\rho s_N(\mathbf{u})u\}, \quad i = 1, ..., N-1. \tag{2.8}$$

(2.8) expresses nothing but a trivial cancellation property in $(\partial_x u)^2$. However, this observation turns out to be crucial for our numerical purposes. Indeed when applied to TW solutions of (2.6), the identities (2.8) can be understood as $(N-1)$ Generalized Rankine-Hugoniot (GRH) conditions (see [19]). To exemplify this claim, let us temporarily adopt the following closure for the viscosity laws : $\mu_i(\mathbf{u}) = \mu_i^0 T_i$ where for each $i = 1, \ldots, N$, μ_i^0 denotes a given positive real number (see also [10] for complementary examples). Let us recall that such a closure can be inferred from the classical transport theory for dilute gases. Then, (2.8) are easily seen to boil down to :

$$\{\partial_t\rho(\frac{\mu_N}{\mu}s_i - \frac{\mu_i}{\mu}s_N(\mathbf{u})) + \partial_x\rho(\frac{\mu_N}{\mu}s_i - \frac{\mu_i}{\mu}s_N(\mathbf{u}))u\} = 0, \quad i = 1, ..., N-1. \tag{2.9}$$

Therefore, (2.9) clearly provide us with $(N-1)$ jump conditions that, with the three classical Rankine-Hugoniot conditions respectively for ρ, ρu and ρE, yield a complete

set of jump relations for defining the endpoints of a TW solution of (2.6). Let us underline that such endpoints clearly depend on the ratios of the viscosities.
More generally, the proportionality in the evolution in time of all the entropy balance equations (as put forward in (2.8)) expresses a well-known property of general convective-diffusive systems in non conservation form. Namely, the endpoints of their TW solutions entirely depend on the shape of the diffusive tensor under consideration (see [42] and [85]). As underlined in the next section, this key property will make the numerical approximation of the TW solutions of (2.8) a non classical issue (see also [87]). In the present work, we show how to enforce numerically for validity the GRH relations (2.8).

2.2 Numerical approximation

In this section, we present a suitable numerical method for approximating the solutions of (2.1). This method can be understood as a prediction-correction strategy. In the prediction step, we just propose to solve the equivalent form (2.6) with appropriate initial data. Such an approach sounds quite natural but however will be seen to fail in preserving the validity of the GRH conditions at the discrete level. Highlighting the roots of such a failure will naturally motivate the correction step which precisely aims at restoring the validity of (2.8) at each time step.
Let be given a constant time step Δt and a constant space step Δx. Introducing $x_{j+1/2} = (j+1/2)\Delta x$ for $j \in Z$ and $t^n = n\Delta t$ for $n \in N$, the approximate solution of (2.6), say $\mathbf{u}_\lambda(x,t)$, with $\mathbf{u}_0$ as initial data is classically sought as a piecewise constant function on each slab $C_j^n = [x_{j-1/2}; x_{j+1/2}) \times [t^n; t^{n+1})$ for $(n,j) \in N \times Z$, and we set

$$\mathbf{u}_\lambda(x,t) = \mathbf{u}_j^n \quad \text{for all} \quad (x,t) \in C_j^n, \quad j \in Z, \quad n \in N.$$

Here λ stands for the ratio $\lambda = \Delta t/\Delta x$, and for completeness we set

$$\mathbf{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x)dx.$$

Assuming that the approximate solution $\mathbf{u}_\lambda(x,t^n)$ is known, we propose to advance it to the next time level t^{n+1} according to a classical splitting stategy between the convective and diffusive parts of (2.6). We now describe these two steps into details.

First step ($t^n \rightarrow t^{n+1=}$) This step consists in solving the first order underlying system in (2.6), that is to say:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \tag{2.10}$$

with $\mathbf{u}_\lambda(x,t^n)$ as initial data, and supplemented with the following Lax entropy inequality :

$$\partial_t \rho s_N(\mathbf{u}) + \partial_x \rho s_N(\mathbf{u})u \leq 0. \tag{2.11}$$

Let us underline that (2.11) is actually needed for uniqueness of the weak solutions of (2.10). Generically and for any given $\mathbf{u}_L$ and $\mathbf{u}_R$ in the phase space, we then denote by $\mathbf{u}(.,\mathbf{u}_L,\mathbf{u}_R)$ the self similar solution of the Riemann problem (2.10)-(2.11) with $\mathbf{u}(x,0) = \mathbf{u}_L$ if $x < 0$ and $\mathbf{u}(x,0) = \mathbf{u}_R$ if $x > 0$ as initial data.

To define $\mathbf{u_j^{n+1=}}$, we propose to use the celebrated Godunov method under the usual CFL condition

$$\lambda \max_{\mathbf{u}}(|\lambda_i(\mathbf{u})|, i = 1, ..., N+2) \leq \frac{1}{2}, \tag{2.12}$$

where the maximum is taken over all the $\mathbf{u}$ under consideration. As a consequence, it is well known that $\mathbf{u}_j^{n+1=}$ can be given an updating formula in full conservation form and obeys in addition a discrete cell entropy inequality. More precisely we have:

Theorem 4

Under the CFL condition (2.12), the Godunov method applied to (2.10)-(2.11) with $\mathbf{u}_\lambda(x, t^n)$ *as initial data leads to*

$$\mathbf{u}_j^{n+1=} = \mathbf{u}_j^n - \lambda \Delta \mathbf{f}_{j+1/2}^n, \quad \Delta \mathbf{f}_{j+1/2}^n = \mathbf{f}_{j+1/2}^n - \mathbf{f}_{j-1/2}^n, \quad j \in Z, \tag{2.13}$$

where we have classically set $\mathbf{f}_{j+1/2}^n = \mathbf{f}(\mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n))$ *for all* $j \in Z$.

In addition, the following cell entropy inequality holds true:

$$(\rho s_N)_j^{n+1=} := \rho s_N(\mathbf{u}_j^{n+1=}) \leq (\rho s_N)_j^n - \lambda \Delta \{\rho s_N u\}_{j+1/2}^n, \tag{2.14}$$

with $\{\rho s_N u\}_{j+1/2}^n = \{\rho s_N u\}(\mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n))$, *as well as the maximum principles on the specific entropies*

$$(s_i)_j^{n+1=} \leq \max((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad i = 1, ..., N. \tag{2.15}$$

Let us recall that the inequality (2.14) comes from the entropy selection principle (2.11) together with the Jensen inequality associated with the averaging procedure in the Godunov method. As a consequence, such an inequality is strict generally speaking.

To conclude this first step, observe that one could have used a suitable approximate Riemann solver instead of the exact one for solving the Cauchy problem (2.10)-(2.11). Indeed, the forthcoming analysis entirely stays valid provided that natural stability properties (like consistency, average conservation, average entropy dissipation, ...) are assumed to hold true for the approximate Riemann solver under consideration. We refer the reader to [19] for such considerations. Let us now address the second step.

Second step ($t^{n+1=} \to t^{n+1-}$) This step takes into account the diffusive part of (2.6). It amounts to solve

$$\partial_t \mathbf{u} = \mathcal{R}(\mathbf{u}, \partial_x(\mathcal{B}(\mathbf{u})\partial_x \mathbf{u})), \tag{2.16}$$

with $\mathbf{u}_\lambda(x, t^{n+1=})$ as initial data. Notice that the smooth solutions of this system automatically obey the following additional equation:

$$\partial_t \rho s_N(\mathbf{u}) = -\frac{\mu_N}{T_N}(\partial_x u)^2. \tag{2.17}$$

Several numerical techniques are at hand in the litterature to provide us with an updated solution, say $\mathbf{u}_\lambda(x, t^{n+1-})$, which satisfies (2.16) as well as (2.17) in the sense of the finite differences. Here and after [10], we propose the following implicit discretization, to avoid a too restrictive CFL-like condition:

$$\begin{cases} \rho_j^{n+1-} = \rho_j^{n+1=}, \\ (\rho u)_j^{n+1-} = (\rho u)_j^{n+1=} + \Delta t \overline{\partial_x(\mu \partial_x u)}_j^{n+1-}, \\ (\rho E)_j^{n+1-} = (\rho E)_j^{n+1=} + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \\ (\rho s_i)_j^{n+1-} = (\rho s_i)_j^{n+1=} - \Delta t \overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-}, \quad i = 1, ..., N-1. \end{cases} \tag{2.18}$$

Here the finite differences formulae read:

$$\left\{\begin{array}{lcl}\overline{\partial_x(\mu\partial_x u)}_j^{n+1-} & = & \frac{1}{\Delta x^2}(\mu^n_{j+1/2}(M^n u_{j+1}-M^n u_j)-\mu^n_{j-1/2}(M^n u_j - M^n u_{j-1})),\\ \overline{\partial_x(\mu u\partial_x u)}_j^{n+1-} & = & \frac{1}{2\Delta x^2}(\mu^n_{j+1/2}((M^n u_{j+1})^2-(M^n u_j)^2)-\mu^n_{j-1/2}((M^n u_j)^2-(M^n u_{j-1})^2)),\\ \overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-} & = & \frac{1}{\overline{T_{ij}}^{n+1-}}((\mu_i)^n_{j+1/2}(M^n u_{j+1}-M^n u_j)^2+(\mu_i)^n_{j-1/2}(M^n u_j - M^n u_{j-1})^2),\end{array}\right.$$

where $M^n X = (X^{n+1-}+X^{n+1=})/2$ and for all i from 1 to N:

$$\overline{T_{ij}}^{n+1-} = \left\{\begin{array}{ll} -\frac{\varepsilon_i(\tau_j^{n+1-},(s_i)_j^{n+1-})-\varepsilon_i(\tau_j^{n+1-},(s_i)_j^{n+1=})}{(s_i)_j^{n+1-}-(s_i)_j^{n+1=}} & \text{if} \quad (s_i)_j^{n+1-}\neq(s_i)_j^{n+1=},\\ T_i(\tau_j^{n+1-},(s_i)_j^{n+1-}) & \text{otherwise.}\end{array}\right.$$

Let us emphasize that since the density is kept unchanged, solving (2.18) just amounts to solve a linear problem in $M^n u$, with a positive definite symmetric matrix. The unknowns $(\rho E)_j^{n+1-}$ and $\{(\rho s_i)_j^{n+1-}\}_{i=1,\dots,N-1}$ then follow with explicit formulae. We do not discuss this step into details and refer the reader to [10] for more informations as well as for the proof of the following statement.

Lemma 9
The updating formulae (2.18) lead to

$$(\rho s_N)_j^{n+1-} := \{\rho s_N\}(\mathbf{u}_j^{n+1-}) = (\rho s_N)_j^{n+1=} - \Delta t\overline{\frac{\mu_N}{T_N}(\partial_x u)^2}_j^{n+1-}. \tag{2.19}$$

Moreover, the following maximum principles hold true:

$$(s_i)_j^{n+1-} \le (s_i)_j^{n+1=} \le \max((s_i)^n_{j-1},(s_i)^n_j,(s_i)^n_{j+1}), \quad i=1,\dots,N. \tag{2.20}$$

Thus, at the end of this second step, (2.16) and (2.17) can be given a consistent definition in the sense of finite differences. To summarize the prediction step (*i.e.* the first step and the second step), we have with clear notations

$$\begin{array}{rcl}\rho_j^{n+1-} & = & \rho_j^n - \lambda\Delta\{\rho u\}^n_{j+1/2}\\ (\rho u)_j^{n+1-} & = & (\rho u)_j^n - \lambda\Delta\{\rho u^2+\sum_{i=1}^N p_i\}^n_{j+1/2} + \Delta t\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}\\ (\rho E)_j^{n+1-} & = & (\rho E)_j^n - \lambda\Delta\{(\rho u^2+\sum_{i=1}^N p_i)u\}^n_{j+1/2} + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-}\\ (\rho s_i)_j^{n+1-} & = & (\rho s_i)_j^n - \lambda\Delta\{\rho s_i u\}^n_{j+1/2} - \Delta t\overline{\frac{\mu_i}{T_i}(\partial_x u)^2}_j^{n+1-}, \quad i=1,\dots,N-1,\end{array}$$

$$(\rho s_N)_j^{n+1-} \le (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}^n_{j+1/2} - \Delta t\overline{\frac{\mu_N}{T_N}(\partial_x u)^2}_j^{n+1-}, \tag{2.21}$$

where as a consequence of (2.14), the inequality is strict generally speaking. The classical approach would consist in choosing $\mathbf{u}_j^{n+1} = \mathbf{u}_j^{n+1-}$. But, because of the strict inequality in (2.21) the cancellation property, responsible for (2.8), is lost and the GRH conditions can no longer be satisfied. Obviously, such a failure persists when using a suitable approximate Riemann solver for (2.10)–(2.11). In the next section, numerical evidences will show that the corresponding approximate solutions grossly disagree with the exact ones. This discussion then clearly motivates the need for a correction step to define the updated solution $\mathbf{u}^h(x,t^{n+1})$ so that the required GRH conditions hold. We propose a correction step which extends the procedure introduced in [10] to arbitrary $N \geq 2$ general pressure laws and we prove its relevance.

Correction step ($t^{n+1-} \to t^{n+1}$) We keep unchanged the updated conservative variables, i.e. :

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad (\rho E)_j^{n+1} = (\rho E)_j^{n+1-},$$

and we propose to define the N entropies $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ as the solutions of the following non linear algebraic problem :

$$\begin{gathered} \overline{\mu_i}\overline{T_N}_j^{n+1-}(\{\rho s_N\}(\mathbf{u}_j^{n+1}) - (\rho s_N)_j^n + \lambda\Delta\{\rho s_N u\}_{j+1/2}^n) \\ = \\ \overline{\mu_N}\overline{T_i}_j^{n+1-}((\rho s_i)_j^{n+1} - (\rho s_i)_j^n + \lambda\Delta\{\rho s_i u\}_{j+1/2}^n), \\ i = 1, ..., N-1. \end{gathered} \tag{2.22}$$

Here $\overline{\mu_i}$ is given a natural consistent definition for all $i = 1, ..., N$. Notice that (2.22) is nothing but a discrete version of the GRH conditions (2.8). The numerical fluxes and the definitions of $\overline{T_i}_j^{n+1-}$ entering (2.22) are directly inherited from the prediction step. The following result assesses the relevance of the correction step.

Theorem 5

The algebraic problem (2.22) admits a single solution $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ which obeys in addition the following entropy inequalities

$$\begin{aligned} &(\rho s_i)_j^{n+1} \leq (\rho s_i)_j^n - \lambda\Delta\{\rho s_i u\}_{j+1/2}^n, \quad i = 1, ..., N-1, \\ &(\rho s_N)_j^{n+1} := \{\rho s_N\}(\mathbf{u}_j^{n+1}) \leq (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}_{j+1/2}^n, \end{aligned} \tag{2.23}$$

as well as the maximum principles on the specific entropies

$$(s_i)_j^{n+1} \leq \max((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad i = 1, ..., N, \quad j \in Z. \tag{2.24}$$

Proof

Let us introduce the following convenient notations

$$(\rho s_N)_j^\star = (\rho s_N)_j^n - \lambda\Delta\{\rho s_N u\}_{j+1/2}^n (\neq \{\rho s_N\}(\mathbf{u}_j^{n+1=})),$$

$$\Gamma_{m,n} = \overline{\mu_m}\overline{T_n}_j^{n+1-} / \overline{\mu_n}\overline{T_m}_j^{n+1-} \geq 0, \quad m = 1, ..., N, \quad , n = 1, ..., N.$$

Let us observe that these quantities are explicitly known at this stage. The first step of the proof consists in recasting (2.22) as follows:

$$\begin{gathered} (\rho s_i)_j^{n+1} = \rho s_i((\rho s_{N-1})_j^{n+1}) = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}((\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}), \\ i = 1, ..., N-2, \\ \\ \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{\rho s_i((\rho s_{N-1})_j^{n+1})\}_{i=1,...,N-2}, (\rho s_{N-1})_j^{n+1}) - (\rho s_N)_j^\star \\ = \\ \Gamma_{N,N-1}((\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}). \end{gathered} \tag{2.25}$$

These identities express the property that the unknowns $(\rho s_i)_j^{n+1}$ for $i = 1, ..., N-2$ can be parametrized by $(\rho s_{N-1})_j^{n+1}$ so that the algebraic problem (2.22) can be solved with $(\rho s_{N-1})_j^{n+1}$ as the only scalar unknown. Thus, existence and uniqueness of the $\{(\rho s_i)_j^{n+1}\}_{i=1,...,N-1}$ as the solutions of (2.25) is equivalent to prove the existence of a non positive real number $\mathcal{M}_j \leq 0$ (namely $\mathcal{M}_j = (\rho s_{N-1})_j^{n+1} - (\rho s_{N-1})_j^{n+1=}$) such that

$$(\rho s_i)_j^{n+1} = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j, \quad i = 1, ..., N-1,$$

$$f(\mathcal{M}_j) = \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{(\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j\}_{i=1,...,N-1}) - \\ \Gamma_{N,N-1}\mathcal{M}_j - (\rho s_N)_j^{\star} = 0. \tag{2.26}$$

Now, since $\partial_{\rho s_i}\{\rho s_N\} = -T_i/T_N < 0$, the mapping $\mathbf{u} \to \rho s_N(\mathbf{u})$ is strictly decreasing with respect to each of the entropies $\{\rho s_i\}_{i=1,...,N-1}$, so that the mapping $\mathcal{M} \to f(\mathcal{M})$ is also strictly decreasing thanks to the positivity of $\Gamma_{m,n}$. Arguing of the asymptotic Weyl's assumptions, let us prove that $\{\rho s_N\}(\mathcal{M}_j) \to +\infty$ (*i.e.* $f(\mathcal{M}_j) \to +\infty$) for a first sufficiently large negative value of $\mathcal{M}_j$, say $\overline{\mathcal{M}_j}$. In that aim, notice that

$$\rho s_i(\mathcal{M}_j) := (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j$$

approaches 0^+ for negative enough values of $\mathcal{M}_j$ for each $i \in \{1, ..., N-1\}$. By the asymptotic condition

$$\lim_{\rho s_i \to 0^+} \rho\varepsilon_i(\rho, \rho s_i) = +\infty,$$

there exists necessarily a first value of $\mathcal{M}_j$, namely $\overline{\mathcal{M}_j}$, solution of

$$(\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} = \sum_{i=1}^{N-1} \rho\varepsilon_i(\rho_j^{n+1}, \rho s_i(\mathcal{M}_j)),$$

that is to say that $\rho\varepsilon_N(\mathcal{M}_j) = 0$ (note that $\rho\varepsilon_N(0) = \rho\varepsilon_N(\rho_j^{n+1=}, (\rho s_N)_j^{n+1=}) > 0$.) But the asymptotic condition

$$\lim_{\rho\varepsilon_N \to 0^+} \rho s_N(\rho, \rho\varepsilon_N) = +\infty$$

just says that

$$\lim_{\mathcal{M}_j \to \overline{\mathcal{M}_j}^+} \rho s_N(\mathcal{M}_j) = +\infty.$$

Note that we have proved in passing that necessarily $\rho s_i(\mathcal{M}_j) > 0$ for all $\overline{\mathcal{M}_j} < \mathcal{M}_j \leq 0$ (since otherwise $f(\mathcal{M}_j) = +\infty$).
Let us next observe the estimate:

$$\begin{aligned} f(0) &= \{\rho s_N\}(\rho_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) - (\rho s_N)_j^{\star} \\ &= \{\rho s_N\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, (\rho E)_j^{n+1-}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) - (\rho s_N)_j^{\star} \leq 0, \end{aligned}$$

as an immediate consequence of (2.21): indeed, notice that thanks to (2.20) and the decreasing property of $\{\rho s_N\}$ with respect to ρs_i, the following inequality holds:

$$\{\rho s_N\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, (\rho E)_j^{n+1-}, \{(\rho s_i)_j^{n+1=}\}_{i=1,...,N-1}) \leq \{\rho s_N\}(\mathbf{u}_j^{n+1-}).$$

Since $f(0) \leq 0$ with $\lim_{\mathcal{M}_j \to \overline{\mathcal{M}_j}^+} f(\mathcal{M}_j) = +\infty$, we necessarily get from the strict monotonicity of the continuous function f the existence of an unique $\mathcal{M}_j$, solution of (2.26), with the property that

$$\overline{\mathcal{M}_j} < \mathcal{M}_j \leq 0,$$

so that necessarily:

$$(\rho s_i)_j^{n+1} = (\rho s_i)_j^{n+1=} + \Gamma_{i,N-1}\mathcal{M}_j > 0, \quad i = 1, ..., N-1.$$

Turning now establishing the maximum principles (2.24), the entropy inequalities we have just obtained rewrite $(\rho s_i)_j^{n+1} \leq (\rho s_i)_j^{n+1=}$ that is to say $(s_i)_j^{n+1} \leq (s_i)_j^{n+1=}$ for all $i = 1, ..., N-1$ since $\rho_j^{n+1} = \rho_j^{n+1=}$. Invoking now (2.15), we obtain the required conclusion for $i = 1, ..., N-1$. Concerning now ρs_N, we have just proved $(\rho s_N)_j^{n+1} \leq (\rho s_N)_j^{\star}$ (but $(s_N)_j^{n+1} \geq (s_N)_j^{n+1=}$), which is classically sufficient to obtain the required maximum principle in the Godunov method setting. This concludes the proof.

2.3 Numerical experiments

In this section, we give some numerical evidences to illustrate the relevance of the prediction-correction scheme we have proposed. To that purpose, we deal with N internal energies of polytropic ideal gases (thermally and calorically perfect) associated with N constant adiabatic exponents $\gamma_i > 1$ for all i=1,...,N. Hence, the total energy reads $\rho E = (\rho u)^2/2\rho + \sum_{i=1}^{N} p_i/(\gamma_i - 1)$. For simplicity we consider constant viscosity laws with a Reynolds number equal to 10^5.
We consider as initial data two constant states, called left state and right state in the following, separated by a discontinuity at $x = 0$ and we approximate the solution on a uniform grid with $\Delta x = 1/300$.

Experiment 1 We set $N = 3$, $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.6, 1.8)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$, while the left and right states of the initial data read: $(\rho, u, p_1, p_2, p_3)_{left} = (4., 1., 1.2, 1.4, 1.6)$, $(\rho, u, p_1, p_2, p_3)_{right} = (2.5568, -1.4305, 0.5162, 0.5103, 0.4999)$.

Experiment 2 We set $N = 3$, $(\gamma_1, \gamma_2, \gamma_3) = (1.4, 1.4, 1.4)$ and $(\mu_2/\mu_1, \mu_3/\mu_1) = (1., 1.)$ while the left and right states of the initial data read: $(\rho, u, p_1, p_2, p_3)_{left} = (3., 1., 1., 1.2, 1.4)$, $(\rho, u, p_1, p_2, p_3)_{right} = (2.6529, -1.1153, 0.8160, 0.9844, 1.1528)$.

In the Figures 2.1 and 2.2, we compare some of the corresponding pressure profiles for the exact solutions (see [19] for the derivation of exact solutions) together with the numerical solutions generated by the scheme both with and without the correction step. As expected, we observe that the classical approach (without correction) fails in capturing the correct solution while the correction step provides us with a numerical solution in good agreement with the exact one. We refer the reader to [19] for additional numerical experiments.

To conclude this paper, let us emphasize that several distinct methods have been actually developed and analysed (see [19]). Their design principle is always focused at preserving the validity of the GRH conditions we have put forward in (2.8). In

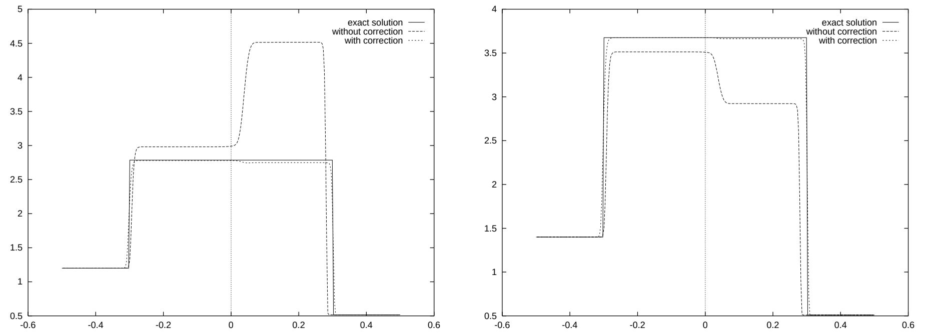

FIG. 2.1 – *Experiment 1 - Pressures 1 and 2*

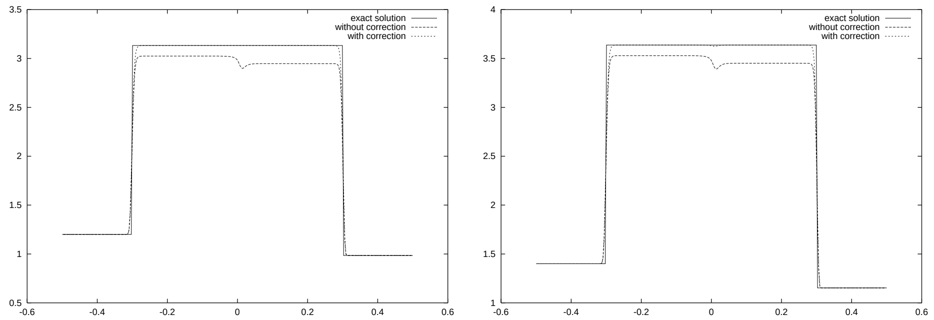

FIG. 2.2 – *Experiment 2 - Pressures 2 and 3*

addition, they all satisfy the same stability estimates but they come with a different cost of numerical evaluation. Here, we have restricted ourselves to the most natural one (along the lines of the first section) but at the expense of the largest numerical cost (we have to solve the nonlinear problem (2.25)). In [19], fully linear correction steps are proposed. In addition, invoking the rotational invariance of the equations under consideration, these methods have been extended to problems with two space variables considering realistic viscosity laws and boundary conditions. We refer the reader to [19].