Transport-Equilibrium Schemes for Computing Nonclassical Shocks

Schémas Transport-Equilibre pour l'Approximation des Chocs Nonclassiques

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Abstract

This paper presents a very efficient numerical strategy for computing weak solutions of a scalar conservation law which fails to be genuinely nonlinear. In such a situation, the dynamics of shock solutions turns out to be mainly driven by a prescribed *kinetic function* that imposes the speed of propagation of the discontinuities. We show how to enforce the validity of the kinetic criterion at the discrete level. The resulting scheme provides in addition sharp profiles. Numerical evidences are included.

Résumé

Ce papier présente un algorithme très efficace pour le calcul des solutions faibles d'une loi de conservation scalaire non vraiment nonlinéaire. Dans ce contexte, la dynamique des solutions choc repose principalement sur la donnée d'une *fonction cinétique* qui fixe la vitesse de propagation des discontinuités. Nous montrons comment forcer la validité du critère cinétique au niveau discret. Le schéma obtenu fournit par ailleurs des discontinuités sans diffusion numérique. Des résultats numériques sont présentés.

1. Introduction

We are interested in computing nonclassical weak solutions of an initial-value problem for a scalar conservation law of the form

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \;\; u(x,t) \in \mathbb{R}, \;\; (x,t) \in \mathbb{R} \times \mathbb{R}^{+*}, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

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where $f : \mathbb{R} \to \mathbb{R}$ is a (smooth) *nonconvex* flux-function. Generally speaking, solutions of problem (1) may be discontinuous and are not uniquely determined by initial data u_0. According to a general regularization principle, we thus ask solutions of (1) to satisfy a single entropy inequality of the form

$$\partial_t U(u) + \partial_x F(u) \leq 0, \tag{2}$$

where $U : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are *specified* functions such that U is strictly convex and $F' = U'f'$. When f is convex, entropy condition (2) actually selects a unique *classical* solution of (1). When f fails to be convex, it is necessary to supplement (1)-(2) with an additional selection criterion called *kinetic relation* from [3]. More precisely, the Riemann problem associated with (1)-(2) still admits a one-parameter family of solutions, which may contain shock waves violating Lax shock inequalities. Such discontinuities are referred as to *undercompressive shocks* or *nonclassical shocks*. In order for the uniqueness to be ensured, a *kinetic relation* needs to be added along each nonclassical discontinuity connecting a left state u_- to a right state u_+. It takes the form $u_+ = \varphi^\flat(u_-)$ or $u_- = \varphi^{-\flat}(u_+)$ where $\varphi^\flat$ is the so-called *kinetic function* and $\varphi^{-\flat}$ its inverse. Then, the speed of propagation is $\sigma(u_-, \varphi^\flat(u_-)) = \left[f(\varphi^\flat(u_-)) - f(u_-)\right] / \left[\varphi^\flat(u_-) - u_-\right]$ by Rankine-Hugoniot conditions. We refer to [3] for a general theory of nonclassical entropy solutions.

The numerical approximation of nonclassical solutions is known to be very challenging and still constitutes an open problem nowadays. The main difficulty is the respect of the kinetic relation at the discrete level. In this paper, we present a new scheme for capturing discontinuities whose dynamics is driven by a kinetic function. Our strategy deals directly with the kinetic function $\varphi^\flat$ to tackle the nonclassical solutions. The resulting algorithm provides numerical results in full agreement with exact ones, whatever the strength of the shocks are. In particular, our scheme leaves sharp isolated nonclassical shocks.

2. The case of cubic flux and nonclassical Riemann solver

Without loss of generality, we take $f(u) = u^3$ which is to some extent the simplest example of a nonconvex function, and refer to [1] for more general flux functions. We consider weak solutions of (1) satisfying entropy inequality (2) with $U(u) = u^2$ and $F(u) = \frac{3}{4}u^4$, and choose (again without restriction)

$$\varphi^\flat(u) = -\beta\, u, \tag{3}$$

as a kinetic function, with $\beta \in [1/2, 1)$ so that each nonclassical shock obeys the entropy inequality (2). We also define $\varphi^\sharp(u) = -u - \varphi^\flat(u) = (\beta - 1)u$.

Given two constant states u_l, u_r such that $u_l > 0$, we now consider a Riemann initial data u_0 defined by $u_0(x) = u_l$ if $x < 0$ and $u_0(x) = u_r$ if $x > 0$. Following [3], the *nonclassical Riemann solver* associated with (1)-(2)-(3) is given as follows :

(1) If $u_r \geq u_l$, the solution is a rarefaction wave connecting u_l to u_r.

(2) If $u_r \in [\varphi^\sharp(u_l), u_l)$, the solution is a classical shock wave connecting u_l to u_r.

(3) If $u_r \in (\varphi^\flat(u_l), \varphi^\sharp(u_l))$, the solution contains a nonclassical shock connecting u_l to $\varphi^\flat(u_l)$, followed by a classical shock connecting $\varphi^\flat(u_l)$ to u_r.

(4) If $u_r \leq \varphi^\flat(u_l)$, the solution contains a nonclassical shock connecting u_l to $\varphi^\flat(u_l)$, followed by a rarefaction connecting $\varphi^\flat(u_l)$ to u_r.

3. Numerical approximation

We now present a suitable algorithm for approximating the nonclassical Riemann solutions of previous section. The method relies on the kinetic function $\varphi^\flat$ only, but in no way on the corresponding nonclassical

Riemann solver. It is made of two steps : an equilibrium step and a transport step. In the equilibrium step, we propose to modify any given consistent and conservative scheme for (1) in order to put at stationary equilibrium nonclassical discontinuities. Then, the transport step propagates these discontinuities.

Let be given a constant time step Δt and a constant space step Δx. Introducing $x_{j+1/2} = j\Delta x$ for $j \in \mathbb{Z}$ and $t^n = n\Delta t$ for $n \in \mathbb{N}$, we seek at each time t^n an approximation u_j^n of solution u on each interval $C_j = [x_{j-1/2}; x_{j+1/2})$, $j \in \mathbb{Z}$. In this context, we choose a two-point numerical flux function $g : (u, v) \to g(u, v)$ consistent with the flux fonction f and set $\lambda = \Delta t/\Delta x$. We now describe the two steps of our strategy.

First step ($t^n \to t^{n+1-}$) This first step aims at making stationary some of admissible discontinuities of problem (1) (see [1] for motivation). In this paper, we will focus ourselves only on the nonclassical discontinuities, that is on the most difficult discontinuities to capture numerically. It is a matter of discontinuities separating two states u_- and u_+ such that $u_+ = \varphi^\flat(u_-) < \varphi^\sharp(u_-)$ when $u_- > 0$. With this in mind, we introduce the following nonconservative update formula :

$$u_j^{n+1-} = u_j^n - \lambda(g^L_{j+1/2} - g^R_{j-1/2}), \quad j \in \mathbb{Z}, \tag{4}$$

where the numerical fluxes $g^L_{j+1/2}$ and $g^R_{j+1/2}$ are defined as follows when $u_j^n > 0$:

$$g^L_{j+1/2} = \begin{cases} g(u_j^n, \varphi^{-\flat}(u_{j+1}^n)) & \text{if} \quad u_{j+1}^n < \varphi^\sharp(u_j^n), \\ g(u_j^n, u_{j+1}^n) & \text{otherwise}, \end{cases} \tag{5}$$

$$g^R_{j+1/2} = \begin{cases} g(\varphi^\flat(u_j^n), u_{j+1}^n) & \text{if} \quad u_{j+1}^n < \varphi^\sharp(u_j^n), \\ g(u_j^n, u_{j+1}^n) & \text{otherwise}, \end{cases} \tag{6}$$

and, in a first approach at least, coincide with $g_{j+1/2}$ when $u_j^n \le 0$ (see again [1] for details). With these definitions, it is easy to check that discontinuities separating two states u_- and u_+ such that $u_+ = \varphi^\flat(u_-)$ are kept at stationary equilibrium during this first step.

Second step ($t^{n+1-} \to t^{n+1}$) This step is concerned with the transport of discontinuities left stationary during the first step. We first recall that the speed of propagation $\sigma(u_-, u_+)$ of a discontinuity between u_- and u_+ is given by Rankine-Hugoniot conditions, that is $\sigma(u_-, u_+) = [f(u_+) - f(u_-)] \,/\, [u_+ - u_-]$. We then define at each interface $x_{j+1/2}$ a speed of propagation $\sigma_{j+1/2}$ by

$$\sigma_{j+1/2} = \begin{cases} \sigma(u_j^{n+1-}, u_{j+1}^{n+1-}) & \text{if} \quad u_{j+1}^n < \varphi^\sharp(u_j^n), \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

and solve at each discontinuity $x_{j+1/2}$ a transport equation with speed $\sigma_{j+1/2}$. In order to get a new approximation u_j^{n+1} at time $t^{n+1} = t^n + \Delta t$, we propose to pick up randomly on interval $[x_{j-1/2}, x_{j+1/2}[$ a value in the juxtaposition of these Riemann solutions at time Δt chosen small enough to avoid wave interactions. Given a well distributed random sequence (a_n) within interval $(0, 1)$, it amounts to set :

$$u_j^{n+1} = \begin{cases} u_{j-1}^{n+1-} & \text{if} \quad a_{n+1} \in [0, \lambda\sigma^+_{j-1/2}[, \\ u_j^{n+1-} & \text{if} \quad a_{n+1} \in [\lambda\sigma^+_{j-1/2}, 1 + \lambda\sigma^-_{j+1/2}[, \\ u_{j+1}^{n+1-} & \text{if} \quad a_{n+1} \in [1 + \lambda\sigma^-_{j+1/2}, 1[, \end{cases} \tag{8}$$

with $\sigma^+_{j+1/2} = \max(\sigma_{j+1/2}, 0)$ and $\sigma^-_{j+1/2} = \min(\sigma_{j+1/2}, 0)$ for all $j \in \mathbb{Z}$. This achieves the description.

4. Numerical experiments

We propose numerical evidences to illustrate the relevance of the scheme we have proposed. We consider a Roe scheme as a basic numerical flux g, and following Collela [2], we use van der Corput random sequence for (a_n). The flux f is still taken to be $f(u) = u^3$ and concerning the kinetic function $\varphi^\flat$, we set $\beta = \frac{3}{4}$ in (3). We address the two typical nonclassical behaviors of the Riemann solution given in Section 2, when taking $u_l = 4$ and u_r equal to -2 (test 1) and -5 (test 2). Numerical solutions are plotted on Figure 1. The mesh contains 100 points per unit interval and initial discontinuity is located at $x = 0$.

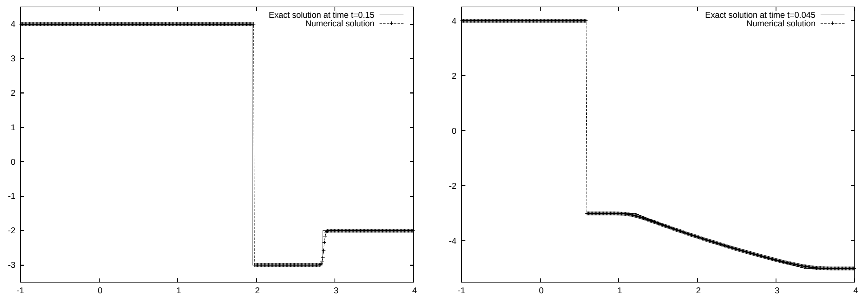

Figure 1. Nonclassical solutions : test 1 (Left) and test 2 (Right)

We observe that numerical solutions fully agree with exact ones. In particular, the left and right states of the nonclassical waves are exactly captured while there are not any points in its profile. The kinetic criterion is respected perfectly, which is remarkable. For test 1, we note that the classical shock contains numerical diffusion induced by the Roe scheme. In [1], we show how to slightly modify the definitions of the numerical fluxes $g^L_{j+1/2}$ and $g^R_{j+1/2}$ in (5)-(6) in order to make sharp the classical shocks, too.

To conclude, we have presented a powerful numerical strategy for computing nonclassical solutions whose dynamics is dictated by a kinetic function. The idea was to modify any given conservative scheme in order to properly capture the undercompressive shocks. Note that this is done *without explicitly using the knowledge of the underlying nonclassical Riemann solver*, contrarily to Glimm's method for instance. In this context and up to our knowledge, our algorithm is the only one providing sharp nonclassical interfaces propagating at the right speed whatever the strengh of interfaces are. A subsequent paper [1] deals with loss of conservation estimates and stability properties of the scheme, together with its extension to the case of systems. Application to pedestrian flows is also under preparation.

References

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