Relaxation and numerical approximation of a two fluid-two pressure model

Annalisa Ambroso* — Christophe Chalons**,***,**** — Frédéric Coquel***,**** — Thomas Galié*

* DEN/DANS/DM2S/SFME/LETR CEA-Saclay, F-91191 Gif-sur-Yvette, France, {annalisa.ambroso, thomas.galie}@cea.fr
** Université Paris Diderot-Paris 7 chalons@math.jussieu.fr
*** UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France,
**** CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France, coquel@ann.jussieu.fr

ABSTRACT. This work is concerned with the numerical approximation of a two fluid-two pressure diphasic model. This model can be seen as two Euler systems coupled by a contact discontinuity. On the basis of a relaxation approach, we propose an approximate Riemann solver that captures exactly these contact discontinuities, is conservative for the mass of each phase and the total momentum, and obeys a $L^1$-stability property.

KEYWORDS: Two fluid-two pressure model, relaxation, Riemann solver, finite volumes.

1. The mathematical model and some basic properties

In recent years, the modelling and computation of two-phase flows by means of a two fluid-two pressure system has gained interest. In this work, we are interested in the numerical approximation of this system which reads, in one space dimension and in the absence of source terms,

$$
\begin{align*}
\partial_t \alpha_1 + u_1 \partial_x \alpha_1 &= 0, \\
\partial_t (\alpha_1 \rho_1) + \partial_x (\alpha_1 \rho_1 u_1) &= 0, \\
\partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - p_1 \partial_x \alpha_1 &= 0, \\
\partial_t (\alpha_2 \rho_2) + \partial_x (\alpha_2 \rho_2 u_2) &= 0, \\
\partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 p_2(\rho_2)) - p_2 \partial_x \alpha_2 &= 0,
\end{align*}
$$

[1]
for $t > 0$ and $x \in \mathbb{R}$. The main unknowns $\alpha_k, \rho_k$ and $u_k$ represent the volume fraction, the density and the velocity of the phase $k = 1, 2$ and we have $\alpha_1 + \alpha_2 = 1$. These are expected to belong to the following phase space

$$\Omega = \{ \mathbf{u} = (\alpha_1, \alpha_1 \rho_1, \alpha_1 \rho_1 u_1, \alpha_2 \rho_2, \alpha_2 \rho_2 u_2) \in \mathbb{R}^5 \text{ such that } \alpha_1 + \alpha_2 = 1, \alpha_k > 0 \text{ and } \rho_k > 0 \text{ for } k = 1, 2 \}.$$ 

The pressure laws $p_k, k = 1, 2$ are given smooth functions such that $p'_k(\rho_k) > 0$, $\lim_{\rho_k \to 0} p_k(\rho_k) = 0$ and $\lim_{\rho_k \to \infty} p_k(\rho_k) = +\infty$. In order to close this system, we set the interfacial velocity $u_I$ and pressure $p_I$ to be

$$u_I = u_2, \quad p_I = p_1(\rho_1).$$

This choice was first proposed in [BAE 86] and we refer the reader to [GAL 04] for a comprehensive study of more general closure laws. The self-similar solutions of this system are studied in [EMB 92]. The first consequence of [2] is that the void fraction will be always satisfied in practice. We are mainly interested in subsonic flows so that the restriction $u_2 \neq u_k \pm c_k, k = 1, 2$ will be always satisfied in practice.

Then, the particular choice [2] allows to give sense to the system even if it cannot be written in a conservative form, as it was shown in [COQ 02]. Indeed, the linear degeneracy property makes the nonconservative products $u_I \partial_x \alpha_k$ and $p_I \partial_x \alpha_k$ locally well defined by finding four $\lambda_0$-Riemann invariants whose gradients are linearly independent. Classical considerations lead to the following parametrisation of the so-called admissible $\lambda_0$-contact discontinuities.

Let us note $h_1$ the enthalpy of the phase 1 defined as $h_1(\rho_1) = e_1(\rho_1) + p_1(\rho_1) \rho_1$, where the internal energy $e_1$ verifies $e'_1(\rho_1) = -\frac{p_1(\rho_1)}{\rho_1^2}$.

**Theorem 1** Let $\mathbf{u}_-$ and $\mathbf{u}_+$ be two constant states in $\Omega$. These states can be joined by an admissible $\lambda_0$-contact discontinuity if and only if the jump relations

$$
\begin{align*}
\begin{cases}
u := u_2_+ = u_2_-, & \\
m := \alpha_1 - \alpha_1 (u_1_+ - u_2) = \alpha_1 + \rho_1^+ (u_1_+ - u_2), & \\
\mu := \alpha_1 (\rho_1^- - \rho_2) = \mu_1^- + \alpha_1 + \rho_1 (\rho_1^+ + \rho_2^+ - \rho_2^+), & \\
\frac{m^2}{2 \alpha_1^- \rho_1^2} + h_1(\rho_1^-) = \frac{m^2}{2 \alpha_1^+ \rho_1^2} + h_1(\rho_1^+), & \\
\end{cases}
\end{align*}
$$

hold true. The speed of propagation of the discontinuity is given by $u_2$. [4]
2. Relaxation approximation

The aim of this section is to propose a suitable relaxation approximation of system [1]. For that, we start from the principle that most of the difficulties arising in the (attempt of) resolution of system [1] are closely related to the nonlinearities of the pressure laws \( p_k \rightsquigarrow p_k(\rho_k) \). In the spirit of [JIN 95], [GOD 00], [CHA 05], we then consider an enlarged system with two additional scalar unknown quantities associated with two linearizations \( \Pi_k \) of the pressure laws \( p_k \). The point is to modify the pressure laws in the convective part of [1] in order to get a quasilinear enlarged system, and to move the nonlinearities in a stiff relaxation term. This relaxation procedure is based on the idea that solutions of the original system [1] are the limit of the solutions of the proposed enlarged system with relaxation source term in the regime of an infinite relaxation coefficient (see for instance [CHA]).

As a relaxation approximation, we propose the following system of balance laws:

\[
\begin{align*}
\partial_t \alpha_1 + u_I \partial_x \alpha_1 &= 0, \\
\partial_t (\alpha_1 p_1) + \partial_x (\alpha_1 \rho_1 u_1) &= 0, \\
\partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 \Pi_1) - \Pi_I \partial_x \alpha_1 &= 0, \\
\partial_t (\alpha_2 p_2) + \partial_x (\alpha_2 \rho_2 u_2) &= 0, \\
\partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 \Pi_2) - \Pi_I \partial_x \alpha_2 &= 0, \\
\partial_t T_1 + u_I \partial_x T_1 &= \lambda(\tau_1 - T_1), \\
\partial_t T_2 + u_I \partial_x T_2 &= \lambda(\tau_2 - T_2),
\end{align*}
\]

for some positive relaxation parameter \( \lambda > 0 \). Here, \( T_k \) and \( \Pi_k \) can be understood as relaxation variables associated with the specific volume \( \tau_k = 1/\rho_k \) and the pressure \( p_k \), and we set

\[
\Pi_k = p_k(1/T_k) + a_k^2(T_k - \tau_k), \quad k = 1, 2,
\]

for some constants \( a_k \) precised just below. These relaxation quantities \( T_k \) and \( \Pi_k \) are expected to converge towards the corresponding equilibrium ones \( \tau_k \) and \( p_k \) in the regime of an infinite relaxation rate \( \lambda (\lambda \rightarrow +\infty) \). However, for that to be true and in order to prevent the relaxation system [5] from instabilities in the regime of large values of \( \lambda (\lambda \gg 1) \), the free parameters \( a_k \) must be such that \( a_k > \rho_k c_k(p_k) \), \( k = 1, 2 \), for all the \( \rho_k \) under consideration. This condition is the so-called Whitham condition (see for instance [CHA 05] and the recent large literature on this subject).

Of course, the proposed interfacial velocity \( u_I \) and pressure \( \Pi_I \) are naturally defined by \( u_I = u_2 \), \( \Pi_I = \Pi_1 \).

The main interest of this relaxation system lies in the fact that the first order system extracted from [5] admits the following five real eigenvalues

\[
\begin{align*}
\lambda_0^R &= u_2, \\
\lambda_1 &= u_1 - a_1 \tau_1, \quad \lambda_2 = u_1 + a_1 \tau_1, \\
\lambda_3 &= u_2 - a_2 \tau_2, \quad \lambda_4 = u_2 + a_2 \tau_2,
\end{align*}
\]

whose associated characteristic fields are all linearly degenerate (the speeds \( \lambda_k^R, k = 1, \ldots, 4 \) are nothing but linearizations of \( \lambda_k, k = 1, \ldots, 4 \)). In addition, system [5] is hyperbolic as soon as \( u_2 \neq \lambda_k^R, k = 1, \ldots, 4 \), in perfect analogy with system [1].
3. Numerical approximation

In this section, we describe a relaxation scheme associated with [5], for approximating the weak solutions of [1]. In what follows, $\Delta t$ and $\Delta x$ denote the time and space steps and $\nu = \frac{\Delta t}{\Delta x}$. The mesh interfaces and intermediate times are $x_{j+1/2} = j\Delta x$, $j \in \mathbb{Z}$ and $t^n = n\Delta t$, $n \in \mathbb{N}$, respectively. As usual in the context of finite volume methods, $u^n_j$ represents an approximate value of the solution in the cells $C_j = [x_{j-1/2}, x_{j+1/2})$. At time $t = 0$, we set $u^n_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx$, $j \in \mathbb{Z}$, where $u_0$ is a prescribed initial condition.

The relaxation method. The numerical strategy is very classical in the context of relaxation methods and proposes to first treat the convective part of [5], and then to take into account the relaxation source term. We begin by defining some approximate values $v^n_j$ at equilibrium for system [5] when setting $v^n_j = (u^n_j, (\tau_1)^n_j, (\tau_2)^n_j)^t$. The two steps are defined as follows.

Step 1 : Evolution in time ($t^n \rightarrow t^{n+1-}$) In this first step, we solve the first order system extracted from [5], that is with $\lambda = 0$, with initial data $v(x, 0) = v^n_j$ if $x \in C_j$, $j \in \mathbb{Z}$. Since this initial condition is piecewise constant, the exact solution is obtained by gluing together the solutions of the Riemann problems set at each interface $x_{j+1/2}$ provided that $\Delta t$ satisfies the usual CFL condition

$$\frac{\Delta t}{\Delta x} \max_{v} \{ |\lambda_i^v(v)|, \ i = 0, \ldots, 5 \} \leq \frac{1}{2}. \tag{7}$$

More precisely

$$v^\nu(x, t) = v^\tau\left(\frac{x - x_{j+1/2}}{t}; v^n_j, v^n_{j+1}\right) \text{ for all } (x, t) \in ]x_j, x_{j+1}[, [0, \Delta t], \tag{8}$$

where $(x, t) \mapsto v^\tau(\frac{x}{\Delta t}; v_L, v_R)$ denotes the self-similar solution of the Riemann problem associated with [5] (with $\lambda = 0$), that is with initial condition

$$v(x, 0) = \begin{cases} v_L & \text{if } x < 0, \\ v_R & \text{if } x > 0. \end{cases} \tag{9}$$

Moreover, the constants $\alpha_k$ are chosen large enough to verify a discretized Whitham condition. Unfortunately, and despite that all the characteristic fields associated with [5] are linearly degenerate, the Riemann solution is difficult to obtain. In particular, it is not given by an explicit formula. Therefore, an approximate solution $(x, t) \mapsto \tilde{v}^\tau(\frac{x}{\Delta t}; v_L, v_R)$ is going to be used in practice. This point will be discussed in the next paragraph.

We then propose a classical averaging procedure to define the sequence $(v^{n+1-})_j$:

$$v^{n+1-}_j = \left(u^{n+1-}_j, (\tilde{T}_1)^{n+1-}_j, (\tilde{T}_2)^{n+1-}_j\right)^t = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{v}^\nu(x, \Delta t)dx. \tag{10}$$

Step 2 : Relaxation ($t^{n+1-} \rightarrow t^{n+1}$) In this second step, we propose to account for the stiff relaxation source term when solving the ordinary differential equations

$$\partial_t v = \lambda R(v), \ x \in \mathbb{R}, \tag{11}$$

where $\lambda$ represents a relaxation parameter.
with \( \mathcal{R}(v) = (0, 0, 0, 0, \tau_1 - T_1, \tau_2 - T_2)^t \), in the asymptotic regime \( \lambda \to \infty \). As initial condition, we take \( v(x, 0) = v_j^{n+1} \) if \( x \in C_j \), \( j \in \mathbb{Z} \). This clearly amounts to set \( v_j^{n+1} = (u_j^{n+1}, (T_1)_j^{n+1}, (T_2)_j^{n+1})^t \) with \( u_j^{n+1} = u_j^{n+1-}, (T_1)_j^{n+1} = (\tau_1)_j^{n+1-} \) and \( (T_2)_j^{n+1} = (\tau_2)_j^{n+1-} \). This completes the description of the method.

The Riemann solution \((x, t) \mapsto \tilde{v}_R(\frac{t}{T}; v_L, v_R)\). As mentioned above, the exact Riemann solution of [5] is not explicitly known, except, of course, in the very particular situation \( \alpha_{1L} = \alpha_{1R} \) which leads to two decoupled systems for each phase. In the general case, the value of the nonconservative product \( \Pi_1 \partial_x \alpha_1 \) is difficult to calculate and this makes the resolution of the Riemann problem pretty challenging (see [AMB ] for more details). Note that this nonconservative product acts on the \( \lambda_0 \)-contact discontinuity only. This discontinuity propagates at the speed \( u_2^* \) and the Rankine-Hugoniot jump relation applied to the fifth equation in [5] (with \( \lambda = 0 \)) gives

\[
-u_2^* \alpha_2 \rho_2 u_2 + [\alpha_2 \rho_2 u_2^2 + \alpha_2 \Pi_2] + < \Pi_1 \partial_x \alpha_1 > = 0.
\]

Here the jump of a quantity \( X \) between the right and left states of a discontinuity is noted \([X]\) and \( < \Pi_1 \partial_x \alpha_1 > \) refers to the mass of the nonconservative product. Since \( u_2 \) is continuous and equals \( u_2^* \), this relation yields

\[
< \Pi_1 \partial_x \alpha_1 >= -[\alpha_2 \Pi_2] .
\]

In order to facilitate the resolution of the Riemann problem, we propose not to find the exact value \([\alpha_2 \Pi_2]\), i.e. not to consider it as an unknown, but to guess it \textit{a priori}. In other words, we replace the actual value \([\alpha_2 \Pi_2]\) with an estimation \([\alpha_2 \Pi_2](u_L, u_R)\) depending on the initial states \( u_L \) and \( u_R \). Once this prediction is provided (see below), we are thus led to consider the following system (recall that \( \lambda = 0 \)),

\[
\begin{align*}
\partial_t \alpha_1 + u_j \partial_x \alpha_1 &= 0 , \\
\partial_t (\alpha_1 \rho_1) + \partial_x (\alpha_1 \rho_1 u_1) &= 0 , \\
\partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 \Pi_1) &= -[\alpha_2 \Pi_2](u_L, u_R) \delta_{x-u_2^* t} , \\
\partial_t (\alpha_2 \rho_2) + \partial_x (\alpha_2 \rho_2 u_2) &= 0 , \\
\partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 \Pi_2) &= [\alpha_2 \Pi_2](u_L, u_R) \delta_{x-u_2^* t} , \\
\partial_t T_1 + u_1 \partial_x T_1 &= 0 , \\
\partial_t T_2 + u_2 \partial_x T_2 &= 0 ,
\end{align*}
\]

for \( t > 0, \ x \in \mathbb{R} \). Solving the Riemann problem associated with [12] turns out to be easier. Indeed, we show in [AMB ] that the corresponding self-similar solution may be explicitly obtained after some manipulations on the Rankine-Hugoniot jump relations. The latter are classical on all the waves except for the \( \lambda_0 \)-contact discontinuity, since the right-hand side vanishes in [12]. Across the \( \lambda_0 \)-wave, they read

\[
\begin{align*}
u^*_2 := u_2^- = u_2^+ , \\
m := \alpha_1 - \rho_1 (u_1^- - u_2^*), \\
m u_1^- + \alpha_1 - \Pi_1^- + \alpha_2 - \Pi_2^- = m u_1^+ + \alpha_1 - \Pi_1^+ + \alpha_2 - \Pi_2^+ , \\
\alpha_2 + \Pi_2^+ - \alpha_2 - \Pi_2^- = [\alpha_2 \Pi_2](u_L, u_R). \end{align*}
\]
Let us briefly comment this set of relations, in comparison to [4]. First of all, the first three equations are similar. They express the continuity of $u_2$ and both the mass and momentum conservation. Then, the fourth ones seem to be different but are shown to be equivalent provided that the prediction $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$ is exact (see below the case of a $\lambda_0$-contact discontinuity joining $u_L$ to $u_R$).

To sum up, we are able to prove the following accuracy and stability properties of the proposed relaxation method (see [AMB] for the proof).

**Theorem 2 (The relaxation method)** The proposed relaxation method:

(i) (Conservativity) : is always conservative on $\alpha_k \rho_k$, $k = 1, 2$ and $\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2$. If, in addition, $\alpha_1$ is constant, the method is also conservative on $\alpha_k \rho_k u_k$, $k = 1, 2$.

(ii) ($L^1$ stability) : provides numerical solutions that remain in the phase space $\Omega$ provided that the free parameters $\alpha_k$, $k = 1, 2$ are chosen sufficiently large.

(iii) (Isolated $\lambda_0$-contact discontinuities) : captures exactly the stationary admissible $\lambda_0$-contact discontinuities of the equilibrium system [1], provided that $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$ is chosen as explained below.

**Remark.** Due to the averaging procedure [10], it is usual that only the stationary admissible $\lambda_0$-contact discontinuity may be exactly computed.

To conclude, it remains to precise the way we predict $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$.

**Estimation of $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$.** Recall that this quantity aims at providing a relevant approximation of the actual jump $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$ across the $\lambda_0$-contact discontinuity in the Riemann solution of [5]. For the sake of accuracy, we would like this approximate value to be exact when $u_L$ and $u_R$ can actually be joined by an admissible $\lambda_0$-contact discontinuity, i.e. when $u_\pm = u_L$ and $u_\pm = u_R$ are such that the jump relations [4] are satisfied. With this in mind, we first define a reconstructed subsonic state $\bar{u}_R$ such that $\bar{u}_{1R} = \alpha_{1R}$ and such that it can be joined to $u_L$ by an admissible $\lambda_0$-contact discontinuity. We compute a first estimate of $\overline{[\alpha_2\Pi_2]}(u_L, u_R) = (\alpha_2 \Pi_2)(\bar{u}_R) - (\alpha_2 \Pi_2)(u_L)$. In a symmetric way, we reconstruct a subsonic state $\bar{u}_L$ and compute a second estimate $\overline{[\alpha_2\Pi_2]}(u_L, u_R) = (\alpha_2 \Pi_2)(u_R) - (\alpha_2 \Pi_2)(\bar{u}_L)$. Of course, we note that if $u_L$ and $u_R$ are joined by an admissible $\lambda_0$-contact discontinuity, we have $\bar{u}_L = u_L$ and $\bar{u}_R = u_R$ so that both possibilities coincide with the exact jump $\overline{[\alpha_2\Pi_2]}(u_L, u_R)$. We give then a criterion to chose among these two, namely we chose the estimate corresponding to the situation which is closer, in the sense of the Euclidean distance, to the original Riemann problem initial data. Actually this criterion doesn’t seem to play an important role: in the numerical tests we performed, no difference can be spotted if one estimate is preferred to the other (see Fig 2).

**Remark.** Notice that if $\alpha_{1L} = \alpha_{1R}$ we have by definition $\bar{u}_L = u_R$ and $\bar{u}_R = u_L$, and then $\overline{[\alpha_2\Pi_2]}(u_L, u_R) = 0$ as expected.
4. Numerical experiments

In this section we present two numerical simulations of the two fluid-two pressure diphasic model. Other numerical experiments can be found in [AMB ]. The pressure laws of the phases are given by \( p_k(\rho_k) = A_k \rho_k^{\gamma_k} \) (\( k = 1, 2 \)), where \( \gamma_k \) is the adiabatic coefficient of phase \( k \) and \( A_k \) a constant depending on the entropy of the gas. In this case, coefficient \( A_k \) takes the constant value of \( 10^5 \) for \( k = 1, 2 \) and we choose \( \gamma_1 = 1.4 \) and \( \gamma_2 = 1.2 \). Initial conditions correspond to shock tube test cases:

\[
\mathbf{u}_0(x) = \begin{cases} 
\mathbf{u}_L & \text{if } x < 10 , \\
\mathbf{u}_R & \text{if } x > 10 , 
\end{cases}
\]

In the first simulation, we compare the Riemann solver based on the relaxation approach we propose and a non-conservative version of the Rusanov solver which is explained in detail in [GAL 04] when the left and right states are chosen in a way to verify relations [4] exactly with \( u_2 = 0 \), i.e. they are separated by a pure stationary \( \lambda_0 \)-contact wave. Computations are performed on 200 cells. Results are displayed at final time 0.02s on Figure 1. In agreement with Theorem 2 above, the relaxation solver provides a perfectly sharp discontinuity at \( x = 10m \) while the Rusanov scheme diffuses the profile. In particular, one can easily notice on Figure 1(b) that only the relaxation scheme exactly preserves the stationary profile of velocity \( u_2 \). The second simulation corresponds to a general shock tube test where we compare results obtained using the two different estimates of \( [\alpha_2 \Pi_2] \) presented above. On fine meshes (400 cells here), no difference can be spotted.

Again, we refer the reader to [AMB ] for additional simulations.

![Figure 1. Pure stationary contact discontinuity](image)

**Acknowledgements.** This work falls within the scope of a joint research program on the coupling of multiphase flow models between CEA-Saclay and Laboratoire Jacques-Louis Lions (http://www.ann.jussieu.fr/groupes/cea). It was partially supported by the NEPTUNE project, with financial support from CEA, (Commissariat à l’Énergie Atomique), EDF, (Electricité de France), IRSN (Institut de Radioprotection et de Sûreté Nucléaire) and AREVA-NP.
Figure 2. Shock tube - Different guess of $[\alpha_2 \Pi_2]$

5. References


