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A conditionally linearly stable second-order traffic model derived from a Vlasov kinetic description

Un modèle de trafic du second ordre conditionnellement linéairement stable

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ABSTRACT

A new second-order traffic model is derived from a nonlinear Vlasov type equation with a source term. The homogeneous part of the system is proven to be hyperbolic. Using a vehicle speed relaxation source term the full system appears to be conditionally linearly stable with instabilities in the dense traffic region. The stability condition depends on the choice of the source term and the model parameters. Numerical experiments confirm the analysis. For a class of source terms, the system is unconditionally linearly stable but numerical experiments show the appearance of nonlinear instabilities that evolve into stop-and-go waves in the dense region.

RÉSUMÉ

Dans cette Note, un modèle de trafic du second ordre est construit à partir d'une description cinétique de type Vlasov. La partie homogène de ce système est hyperbolique. Néanmoins, en utilisant un terme source de relaxation de vitesse, le système complet non homogène s'avère être conditionnellement linéairement stable avec une région d'instabilité localisée dans le régime dense. La condition de stabilité linéaire dépend du choix du terme source et des paramètres ouverts du modèle. Les expériences numériques confirment l'analyse théorique. Pour une certaine classe de termes de source, le système est inconditionnellement linéairement stable ; les expérimentations numériques montrent l'apparition d'instabilités non linéaires qui évoluent en ondes « stop-and-go » dans la région de trafic dense.

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1. Introduction

Further to the seminal studies of Greenschield about vehicular traffic in the 1930's, traffic modelling has captured the interest of many researchers. The subject is tackled at different levels of analysis (micro-, meso- or macroscopic) depending on the studied phenomena or the scale of the future applications. Among the fluid-dynamic models, the second-order traffic flow models can be relevant if nonequilibrium in velocity and possibly instabilities for some flow regimes are expected. To address the criticism of the second-order traffic flow models by Daganzo [1] (see also Helbing and Johansson [2]), several 2×2 systems of conservation laws were proposed, see for instance [3–7]. In a recent work by Illner et al. [8], the second order Aw–Rascle [3] model was derived from a kinetic description [9–12] of traffic flow, with specific approximations and closures. This Note also deals with the construction of macroscopic second-order traffic flow models from a formal Vlasov-based kinetic model but with a different acceleration term involving the spatial derivative of the mean velocity. The resulting model has the required hyperbolicity properties regarding the homogeneous part of the system. Considering the full system with the nonlinear relaxation source term, the analysis shows that the linear stability is generally conditional, depending on some model's parameters. These parameters can be designed in order to adjust the width of the instability region. The dense traffic regime appears to be the most sensitive region in terms of instability. This is in accordance with recent works by Coscia [13] and Helbing and Johansson [2], see also Bagnerini et al. [14]. For particular shapes of source terms, the system is unconditionally linearly stable. Moreover, numerical experiments demonstrate the existence of nonlinear instabilities that develop and degenerate into stop-and-go waves in the dense traffic region (see Herty and Illner [15]). The growth rate is observed to be dependent on the relaxation time parameter, so that this parameter can be set in order to reproduce the expected rate of instability. Ongoing work aims at using the present model in the framework of weather-responsive traffic model. Indeed the impact of weather on traffic can be integrated into the model via weather-dependent fundamental diagrams or relaxation times. Among the other perspectives of this research, a hybrid meso–macro system is developed in order to describe the evolution of the time headway distribution into the model.

2. Vlasov type equation for traffic modeling

Let $f = f(x, v, t)$ be the density distribution of vehicles having individual speed v at position x and time t. To model the traffic flow, let us consider the nonlinear Vlasov type equation with a source term

$$\partial_t f + \partial_x(vf) + \partial_v\big(a(v, f, \partial_x f)f\big) = \eta(f)\big(f^{eq} - f\big) \tag{1}$$

where $f^{eq} = f^{eq}(x, v, t)$ is some statistical equilibrium distribution (to define, see later). The quantity $\eta(f) \geqslant 0$ is the relaxation rate to the equilibrium, possibly depending on f itself through its zero-order and first-order moments and the term $a(v, f, \partial_x f)f$ is a vehicle acceleration/braking term. In what follows, we will assume that

$$\big(1, v, v^2, a\big) f \in \mathcal{C}^0\big([0,\infty)\big) \cap L^1(0,\infty), \quad (1, v) f^{eq} \in \mathcal{C}^0\big([0,\infty)\big) \cap L^1(0,\infty)$$

We respectively introduce the density variable ρ defined as zero-order moment of the distribution f and the flow rate q defined as the first-order moment:

$$(\rho, q) = \int_0^\infty (1, v) f(v)\,\mathrm{d}v \tag{2}$$

The mean vehicle speed u will be defined as

$$u = \frac{q}{\rho} = \frac{\int_0^\infty v f(v)\,\mathrm{d}v}{\int_0^\infty f(v)\,\mathrm{d}v}$$

From physical considerations, it is expected that the traffic density variable ρ belongs to the interval $(0, \rho_M]$ where ρ_M is the maximum vehicle density. The vehicle speed variable u is expected to vary between zero and the maximum mean velocity corresponding to free flow, denoted by u_f.

2.1. Fundamental diagram and equilibrium distribution

Usually, from traffic measurements and data analysis, one gets the fundamental diagram of traffic flows that links the vehicle density with the flow rate q^{eq}, or the vehicle density ρ with the mean vehicle speed u^{eq} under statistical equilibrium traffic conditions (see [16] and [17] for example). One can use either $q^{eq}(\rho)$ or $u^{eq}(\rho)$ according to the compatibility formula $q^{eq}(\rho) = \rho u^{eq}(\rho)$. For example the so-called triangular law (see [16]) defines the equilibrium speed as a piecewise linear function of the density

$$u^{eq}(\rho) = \max\left(0, u_f \min\left(1, \frac{(\rho - \rho_M)\rho_c}{(\rho_c - \rho_M)\rho}\right)\right), \quad \rho \in [0, \rho_M] \tag{3}$$

where $\rho_c \in (0, \rho_M)$ is the critical density that determines the boundary between uncongested and congested traffic conditions. Generally, equilibrium speeds $u^{eq}(\rho)$ are designed such that

$$\left(u^{eq}\right)'(\rho) \leqslant 0 \quad \forall \rho \in [0, \rho_M] \tag{4}$$

expressing a loss of speed at increasing density. Condition (4) will be assumed in the sequel of this paper. Regarding the equilibrium density $f^{eq}(x, v, t)$, one could also use estimation tools to determine what is the statistical law that best fits with the measurements. It is assumed that $\int_0^\infty f^{eq}(v)\,\mathrm{d}v = \rho$, meaning that both actual local distribution f and equilibrium distribution f^{eq} share the same zero-order moment. It will be also assumed that the first moment of the equilibrium distribution f^{eq} is compatible with the equilibrium speed given by the fundamental diagram $\rho \mapsto u^{eq}(\rho)$, i.e.

$$u^{eq}(\rho) = \frac{1}{\rho}\int_0^\infty v f^{eq}(v)\,\mathrm{d}v \tag{5}$$

A closure for f^{eq} for example is to assume its dependency on (x, t) by means of the zero-order moment of f only: $f^{eq}(v; x, t) = f^{eq}(v; \rho(x, t))$. In order to get the expected zero- and first-order moments, $f^{eq}(v; \rho)$ could be searched for example in the form

$$f^{eq}(v; \rho) = \frac{\rho}{u^{eq}(\rho)}\chi\left(\frac{v}{u^{eq}(\rho)}\right), \quad \rho > 0$$

where $\chi \geqslant 0$ is a smooth compactly supported function such that $\int_0^\infty (1, w)\chi(w)\,\mathrm{d}w = (1, 1)$.

2.2. Empirical closure of the acceleration term

The acceleration term captures the acceleration/braking behaviour of the drivers and their reaction to the downstream traffic flow. Because of a lack of information on the underlying stochastic process induced by the behaviour of the drivers, a closure is needed. Integrating Eq. (1) in v over $(0, \infty)$ we get after integrating by parts

$$\partial_t \rho + \partial_x q + \big[a(v, f, \partial_x f) f\big]\big|_{v=0} = 0 \tag{6}$$

If we want to get the expected mass conservation equation $\partial_t \rho + \partial_x q = 0$ that expresses the conservation of the number of vehicles for a lane without on/off-ramps, we should impose the following constraint:

$$\lim_{v \to 0} a(v, f, \partial_x f) f(v) = 0 \tag{7}$$

Let us assume that acceleration/braking driver behaviour is directly linked to the spatial variation of the mean vehicle speed $\partial_x u$: if $\partial_x u > 0$ (resp. if $\partial_x u < 0$), then the driver has to accelerate (resp. slow down) to adapt himself to the main flow. In this respect, the $a(v, f, \partial_x f)$ will be assumed proportional to $\partial_x u$.

A simple way to fulfill the property (7) is to make the acceleration term be proportional to v. We will also assume that the acceleration term is proportional to the mean vehicle spacing, linearly depending on the quantity ρ^{-1}. We then simply close the acceleration term as:

$$a(v, f, \partial_x f) = v\frac{\rho_0}{\rho}\partial_x u \tag{8}$$

where $\rho_0 > 0$ is a constant that has the dimension of a density (vehicles per kilometer). The choice of ρ_0 will be discussed later.

2.3. Closure of the source term

Regarding the source term, we shall consider the following three-parameter form of the relaxation rate function $\eta(f)$:

$$\eta(f) = \eta(\rho, u; \ell, \delta, \alpha) = \left(\frac{|u^{eq}(\rho) - u|}{\ell(\rho)}\right)^{\alpha}\left(\frac{1}{\delta(\rho)}\right)^{1-\alpha} \tag{9}$$

where $\ell(\rho)$ is a characteristic length function, $\delta(\rho)$ is a characteristic time function and $\alpha \in [0, 1]$. The function η has the dimension of the inverse of a time. In addition, we will assume that both $\ell(\rho)$ and $\delta(\rho)$ are smooth, positive, decreasing functions of ρ with

$$\lim_{\rho \to \rho_M} \big\{\ell(\rho), \delta(\rho)\big\} = 0 \tag{10}$$

The assumption (10) indicates that traffic equilibrium is immediately reached in case of traffic jam. In particular, for $\alpha = 0$, the source term is simply a relaxation term towards the equilibrium distribution

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$$\frac{f^{eq}(v;.) - f(v;.)}{\delta(\rho)}$$

whereas for $\alpha = 1$, it becomes

$$\frac{|u^{eq}(\rho) - u|}{\ell(\rho)}\big(f^{eq}(v;.) - f(v;.)\big)$$

i.e. a relaxation term with a rate involving the speed nonequilibrium itself.

2.4. Derivation of macroscopic equations

The requirements above already give the continuity equation

$$\partial_t \rho + \partial_x q = 0 \tag{11}$$

with $q = \rho u$. To close the system, we need an additional equation on q. Let us multiply Eq. (1) by v and integrate over $(0, \infty)$. We get

$$\partial_t q + \partial_x \int_0^\infty v^2 f \, \mathrm{d}v + \int_0^\infty v \partial_v \big[a(v, f, \partial_x f) f\big] \, \mathrm{d}v = \eta(\rho, u)\big(q^{eq}(\rho) - q\big)$$

Integrating by parts, from the assumption (7) we have

$$A = \int_0^\infty v \partial_v [af] \, \mathrm{d}v = -\int_0^\infty a(v, f, \partial_x f) f(v) \, \mathrm{d}v$$

Now using the empirical form (8) we have

$$A = -\frac{\rho_0}{\rho} \partial_x u \int_0^\infty v f(v) \, \mathrm{d}v = -\frac{\rho_0}{\rho} (\partial_x u) q = -\partial_x \left(\rho_0 \frac{u^2}{2}\right)$$

By introducing the pressure variable

$$p = \int_0^\infty (v - u)^2 f \, \mathrm{d}v \tag{12}$$

that acts as a force due to the fluctuations of vehicle speed, we get the momentum equation written in conservation form

$$\partial_t q + \partial_x (qu) - \partial_x \left(\frac{1}{2} \rho_0 u^2\right) + \partial_x p = \eta(\rho, u)\big(q^{eq}(\rho) - q\big) \tag{13}$$

We need a closure model for the pressure. The simplest way is to consider a space invariant pressure meaning that the energy of the fluctuations of vehicle speed is constant (this assumption is also made by Illner et al. [8] to derive the Aw–Rascle system from a Vlasov model). In that case $\partial_x p = 0$ and the resulting second-order traffic model is

$$\partial_t \rho + \partial_x (\rho u) = 0 \tag{14}$$

$$\partial_t (\rho u) + \partial_x \big(\rho u^2\big) - \partial_x \left(\frac{1}{2} \rho_0 u^2\right) = \rho \eta(\rho, u)\big(u^{eq}(\rho) - u\big) \tag{15}$$

3. Hyperbolicity of the homogeneous system

The system of conservation laws is written in vector form

$$\partial_t U + \partial_x F(U) = S(U) \tag{16}$$

with $U = (\rho, \rho u)$. The admissible space for this system is $\Omega^{ad} = \{U = (\rho, \rho u),\ \ \rho \in (0, \rho_M],\ \ u \in [0, u_f]\}$. For smooth solutions, introducing $\tau = \rho^{-1}$, the system can also be written in primitive variable

$$\partial_t \tau + u \partial_x \tau - \tau \partial_x u = 0 \tag{17}$$

$$\partial_t u + (1 - \rho_0 \tau) u \partial_x u = \eta(\rho, u)\big(u^{eq}(\rho) - u\big) \tag{18}$$

i.e. in quasilinear form $\partial_t W + A(W)\partial_x W = S(W)$ where $W = (\tau, u)$ is the vector of primitive variables. In the sequel, we will denote by $c = c(\tau, u)$ the quantity

$$c = \rho_0 \tau u \geqslant 0 \tag{19}$$

which can be seen as a speed of anticipation. The eigenvalues of the system are $\lambda_1(W) = u - c(\tau, u)$ and $\lambda_2(W) = u$. For $U \in \Omega^{ad}$, $c \geqslant 0$ then $\lambda_1(W) \leqslant \lambda_2(W)$. The two eigenvalues are distinct except in the singular case $u = 0$. On the restricted admissible space $\Omega^{ad,\star} = \{U = (\rho, \rho u),\ \rho \in (0, \rho_M],\ u \in (0, u_f]\}$, the homogeneous system is strictly hyperbolic. The respective right eigenvectors are $r_1(W) = (\tau, c)^T$ and $r_2(W) = (1, 0)^T$. It is clear that $\nabla_W \lambda_2(W) \cdot r_2 = 0\ \forall U \in \Omega^{ad,\star}$ so that the 2-field is linearly degenerate (LD). For the 1-field we have

$$\nabla_W \lambda_1(W) \cdot r_1(W) = -(\rho_0 \tau)^2 u \tag{20}$$

so that the characteristic 1-field is genuinely nonlinear (GNL) on $\Omega^{ad,\star}$. From the theory of hyperbolic systems of conservation laws [18], it is then known that, at least for initial data of small amplitude, entropy solutions of Riemann problems exist, made of three constant states separated by a 1-wave which is either a shock wave or a rarefaction fan and a 2-contact discontinuity. In the next sections, we will focus on the linear stability analysis of the full system.

4. Linear stability analysis of the full system

4.1. Case of a relaxation source term with $\alpha = 0$

This case corresponds to $\eta = \delta^{-1}$. We look for plane wave perturbation solutions $\rho = \rho^0 + \tilde{\rho}$, $u = u^0 + \tilde{u}$ of the system linearized towards an equilibrium constant state (ρ^0, u^0), $u^0 = u^{eq}(\rho^0)$:

$$\partial_t \tilde{\rho} + u^0 \partial_x \tilde{\rho} + \rho^0 \partial_x \tilde{u} = 0 \tag{21}$$

$$\partial_t \tilde{u} + \left(1 - \frac{\rho_0}{\rho^0}\right) u^0 \partial_x \tilde{u} = \frac{(u^{eq})'(\rho^0)\tilde{\rho} - \tilde{u}}{\delta(\rho^0)} \tag{22}$$

That means that we are looking for solutions of the form

$$\tilde{\rho} = \tilde{\rho}^0 e^{ikx + \lambda t} \tag{23}$$

$$\tilde{u} = \tilde{u}^0 e^{ikx + \lambda t} \tag{24}$$

where k is a wave number and $\lambda \in \mathbb{C}$. Plane waves will grow during time if $Re(\lambda) > 0$. If $Re(\lambda) < 0$, then perturbations will damp exponentially. If $Re(\lambda) = 0$, then (ρ^0, u^0) will be a center with periodic solutions. For simplicity, let us denote $r = \rho_0/\rho^0 > 0$, $\delta = \delta(\rho^0)$ and $\gamma = (u^{eq})'(\rho^0)$. Notice that $\gamma \leqslant 0$ because of the initial assumption (4). Putting (23), (24) into (21), (22) leads to the eigenvalue problem

$$\begin{pmatrix} iku^0 & ik\rho^0 \\ -\gamma\delta^{-1} & ik(1-r)u^0 + \delta^{-1} \end{pmatrix} \begin{pmatrix} \tilde{\rho}^0 \\ \tilde{u}^0 \end{pmatrix} = -\lambda \begin{pmatrix} \tilde{\rho}^0 \\ \tilde{u}^0 \end{pmatrix} \tag{25}$$

The characteristic polynomial of the eigenvalue system writes

$$P(\lambda) = \lambda^2 + \lambda\left[ik(2-r)u^0 + \delta^{-1}\right] - k^2\left(u^0\right)^2(1-r) + ik\delta^{-1}\left(u^0 + \rho^0\gamma\right)$$

The discriminant Δ of this polynomial of degree 2 is $\Delta = (iku^0 r - \delta^{-1})^2 - 4ik\rho^0\gamma\delta^{-1}$. We have to look for the square roots of this discriminant. For a general complex number z in the form $z = A + iB$, the square roots are given by the formula

$$\sqrt{z} = \sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}} + i\,\mathrm{sgn}(B)\sqrt{\frac{\sqrt{A^2 + B^2} - A}{2}} \tag{26}$$

Here we have $A = \delta^{-2} - k^2(u^0)^2 r^2$ and $B = -2k\delta^{-1}(u^0 r + 2\rho^0\gamma)$. For stability analysis, we only have to deal with the real part of the roots of polynomial $\lambda^\pm$. From the formula (26) we find that

$$Re\left(\lambda^\pm\right) = \frac{1}{2}\left\{-\delta^{-1} \pm \sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}}\right\} \tag{27}$$

The system is linearly stable if and only if $Re(\lambda^-), Re(\lambda^+) \leqslant 0$ for any wave number k. For any $\delta > 0$, it is clear that $Re(\lambda^-) < 0$. The necessary and sufficient condition for $Re(\lambda^+)$ to be negative is

$$\sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}} \leqslant \delta^{-1}$$

A straightforward development gives the condition $\rho^0\gamma^2 + \gamma u^0 r \leqslant 0$ or alternatively

$$\rho^0\gamma + u^0\frac{\rho_0}{\rho^0} \geqslant 0 \tag{28}$$

This condition appears to be independent of the wave number k but also of the relaxation time δ. From the definition of u^0 and γ, we have the following stability condition that depends on the fundamental diagram law $u^{eq}(\rho)$ and on the parameter ρ^0: $-(\rho^0)^2(u^{eq})'(\rho^0) \leqslant \rho_0 u^{eq}(\rho^0)$. As a conclusion, let us state the following theorem:

Theorem 4.1. *The system* (17), (18) *with* $\alpha = 0$ *is conditionally linearly stable according to the choice of the constant* ρ_0. *The necessary and sufficient condition for linear stability towards an equilibrium constant state* $(\rho, u^{eq}(\rho))$ *is*

$$-\rho^2\big(u^{eq}\big)'(\rho) \leqslant \rho_0 u^{eq}(\rho) \tag{29}$$

A sufficient condition for example for (ρ, u) to satisfy (29) is to satisfy

$$-\rho^2\big(u^{eq}\big)'(\rho) \leqslant \left(\frac{\rho}{\rho_M}\right)^2 \rho_0 u^{eq}(\rho)$$

Applying Gronwall's lemma gives the sufficient condition

$$u^{eq}(\rho) \geqslant u^{eq}\big(\rho^\star\big)\exp\left(-\frac{\rho_0}{\rho_M}\frac{\rho-\rho^\star}{\rho_M}\right) \tag{30}$$

for any $\rho^\star \in [0, \rho_M)$. Generally for usual equilibrium laws, the stability condition (29) is not satisfied in the dense traffic region. This is in particular true for the triangular law (3). Assuming $\rho_c = \omega\rho_M$ with $\omega \in [0, 1]$, it is an easy matter of fact to show that the stability condition is

$$u^{eq}(\rho) \geqslant \frac{u_f\omega\rho_M}{(1-\omega)\rho_0} \tag{31}$$

The value of the constant ρ_0 can be calibrated in order to get the expected instability region. For example, if we want to define the stability region by $u^{eq}(\rho) \geqslant \mu u_f$ with $\mu \in (0, 1)$, then ρ_0 is computed as

$$\rho_0 = \frac{\omega}{(1-\omega)\mu}\rho_M \tag{32}$$

4.2. Case $\alpha \neq 0$

For $\alpha \neq 0$, we have

$$\partial_t u + u\left(1 - \frac{\rho}{\rho_0}\right)\partial_x u = \left(\frac{|u^{eq}(\rho) - u|}{\ell(\rho)}\right)^{\alpha}\left(\frac{1}{\delta(\rho)}\right)^{1-\alpha}\big(u^{eq}(\rho) - u\big)$$

In this case, linearizing the equation towards a constant state (ρ^0, u^0), $u^0 = u^{eq}(\rho^0)$ simply gives

$$\partial_t\tilde{u} + u^0\left(1 - \frac{\rho^0}{\rho_0}\right)\partial_x\tilde{u} = 0$$

Plane waves that are solutions of the linearized system must satisfy the compatibility conditions

$$\begin{pmatrix} iku^0 & ik\rho^0 \\ 0 & ik(1-r)u^0 \end{pmatrix}\begin{pmatrix} \tilde{\rho}^0 \\ \tilde{u}^0 \end{pmatrix} = -\lambda\begin{pmatrix} \tilde{\rho}^0 \\ \tilde{u}^0 \end{pmatrix} \tag{33}$$

The eigenvalues are pure imaginary complex numbers and the linearized system is stable towards any constant state $(\rho^0, u^{eq}(\rho^0))$ for any $\rho_0 > 0$.

5. Numerical modeling and experimentation

In this section we build a numerical stable conservative scheme that approximates the solutions of the system (17), (18). A second-order fractional step method allows us to deal with the convective part of the system and the source term separately. For the discretization of the convective part, a Lagrange-plus-remap approach appears suitable and easy to implement. Let us consider a uniform subdivision of the space $\{x_j\}_{j\in\mathbb{Z}}$, $x_j = jh$, where h is the constant space step. From discrete sequences of density $(\rho_j^n)_{j\in\mathbb{Z}}$ and speed $(u_j^n)_{j\in\mathbb{Z}}$ at current instant t^n, we want to compute the sequences at the next time $t^{n+1} = t^n + \Delta t^n$, $\Delta t^n > 0$. The first step of the fractional step method consists in integrating the inhomogeneous part of the system over a time step $\Delta t^n/2$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_j)=0,\qquad \frac{\mathrm{d}}{\mathrm{d}t}(u_j)=\left(\frac{|u^{eq}(\rho_j)-u_j|}{\ell(\rho_j)}\right)^{\alpha}\left(\frac{1}{\delta(\rho_j)}\right)^{1-\alpha}\left(u^{eq}(\rho_j)-u_j\right)$$

with initial conditions $\rho_j(0)=\rho_j^n$, $u_j(0)=u_j^n$. We get $\rho_j^{(1)}$ and $u_j^{(1)}$. The differential problem can be integrated exactly. For example, for $\alpha=0$, we get

$$u_j^{(1)}=u^{eq}\left(\rho_j^n\right)+\left(u_j^n-u^{eq}\left(\rho_j^n\right)\right)\exp\left(-\frac{\Delta t}{2\delta(\rho_j^n)}\right)$$

The second step consists in solving the homogeneous hyperbolic system over a time step Δt. We use a Lagrange-plus-remap approach. The Lagrange substep integrates the equations using a Lagrange formulation. The computational cell I_j initially located at $(x_{j-1/2},x_{j+1/2})$, with $x_{j+1/2}=(j+1/2)h$, is convected according to the vehicle flow. Interface velocities $u_{j+1/2}$ are defined from the structure of the solutions of the Riemann problem at the interfaces. Here, it is natural to consider the upwind interface velocity $u_{j+1/2}=u_{j+1}^{(1)}$. The convected cell number j, after a time step Δt^n has the new size

$$h_j^{(2)}=h+\Delta t^n\left(u_{j+1}^{(1)}-u_j^{(1)}\right) \tag{34}$$

The mass conservation into the cell number j allows us to update the density variable

$$\rho_j^{(2,\star)}=\frac{h}{h_j^{(2)}}\rho_j^{(1)} \tag{35}$$

The equation of the flow rate q in a Lagrangian integral form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{I_t} q(x,t)\,\mathrm{d}x-\int\limits_{I_t}\partial_x\left(\frac{1}{2}\rho_0 u^2\right)\mathrm{d}x=0$$

for any interval I_t moving with the flow. This leads to the following update scheme for the vehicle speeds:

$$u_j^{(2,\star)}=u_j^{(1)}+\frac{\rho_0\Delta t^n}{2h\rho_j^{(1)}}\left(\left(u_{j+1}^{(1)}\right)^2-\left(u_j^{(1)}\right)^2\right) \tag{36}$$

The following CFL-like condition (Courant–Friedrichs–Lewy) forbids the waves interaction between two successive local Riemann problems' solutions:

$$\frac{\Delta t}{h}\left[u_{j-1/2}-\min(0,u_{j+1/2}-c_{j+1/2})\right]\leqslant 1 \tag{37}$$

The remap phase is used to reproject the convected discrete solution onto the initial Eulerian mesh. Introducing the local interface Courant number

$$\nu_{j+1/2}=\frac{\Delta t^n}{h}u_{j+1}^{(1)} \tag{38}$$

then the remap phase simply consists in updating the states using the advance scheme

$$\rho_j^{(2)}=\nu_{j-1/2}\rho_{j-1}^{(2,\star)}+(1-\nu_{j-1/2})\rho_j^{2,\star} \tag{39}$$

$$(\rho_j u_j)^{(2)}=\nu_{j-1/2}(\rho_{j-1}u_{j-1})^{(2,\star)}+(1-\nu_{j-1/2})(\rho_j u_j)^{2,\star},\qquad u_j^{(2)}=\frac{(\rho_j u_j)^{(2)}}{\rho_j^{(2)}} \tag{40}$$

In order to get stability properties, the time step Δt^n must be constrained to satisfy the CFL condition

$$\sup_{j\in\mathbb{Z}}\nu_{j+1/2}\leqslant 1 \tag{41}$$

Notice that using (37), the CFL condition (41) is automatically fulfilled.

The third and last step of the fractional step method consists in integrating the inhomogeneous part of the system once again over a time step $\Delta t^n/2$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_j)=0,\qquad \frac{\mathrm{d}}{\mathrm{d}t}(u_j)=\left(\frac{|u^{eq}(\rho_j)-u_j|}{\ell(\rho_j)}\right)^{\alpha}\left(\frac{1}{\delta(\rho_j)}\right)^{1-\alpha}\left(u^{eq}(\rho_j)-u_j\right)$$

with initial conditions $\rho_j(0)=\rho_j^{(2)}$, $u_j(0)=u_j^{(2)}$. We then get ρ_j^{n+1} and u_j^{n+1}. It can be shown that the whole numerical scheme has very interesting stability and accuracy properties even in the case of stiff sources terms with very small relaxation times.

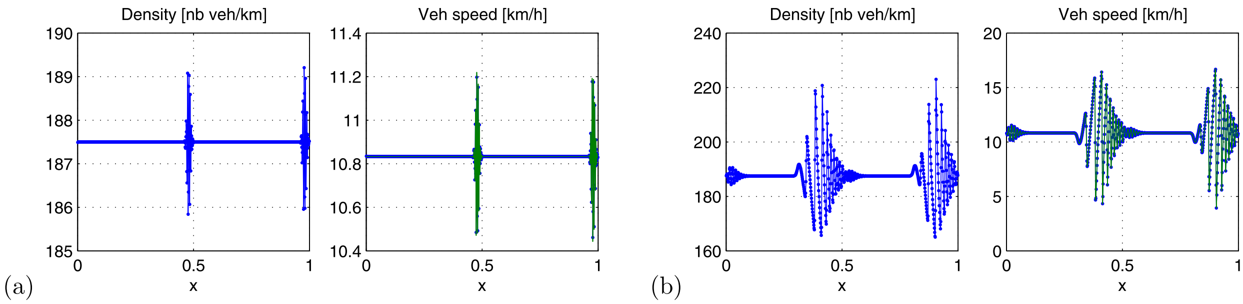

Fig. 1. Growth of linear instabilities in the dense traffic region for $\alpha = 0$. Density and mean speed profiles, (a) at time $t = 0.002$ and (b) at time $t = 0.1$.

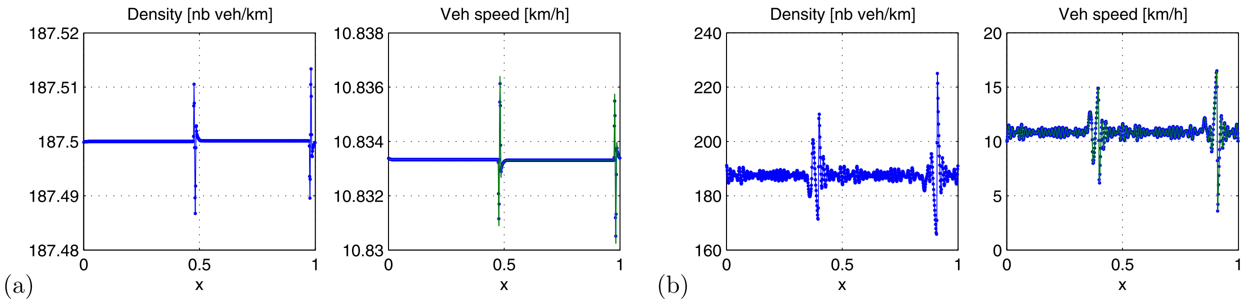

Fig. 2. Growth of nonlinear instabilities in the dense traffic region for $\alpha = 1$. Density and mean speed profiles, (a) at time $t = 0.002$ and (b) at time $t = 0.1$. Instabilities stop to grow after a certain delay.

For numerical experiments, we consider the spatial domain $D = [0, 1]$ with periodic boundary conditions. We use a uniform grid made of 800 points, the CFL number is equal to 0.45 and the time step is computed according to the formula. The "triangular" law (3) is used with the following parameters: $\rho_M = 250$ veh/km, $\rho_c = 50$ veh/km and $u_f = 130$ km/h. For the first test case, the model parameter ρ_0 is computed from the formula

$$\rho_0 = \frac{1}{\mu}\frac{\omega}{1-\omega}\rho_M$$

with $\omega = \frac{\rho_c}{\rho_M} = \frac{1}{5}$ and μ is chosen equal to $\frac{1}{2}$ (linearly unstable region for densities $\rho \in [0, \rho_M]$ such that $u^{eq}(\rho) < \frac{u_f}{2}$). That gives $\rho_0 = \frac{1}{2}\rho_M$. Fig. 1 is designed to show the instability growth in the case $\alpha = 0$. The initial condition consists of a piecewise constant function $\rho^0(x)$ with $\rho^0(x) = \frac{3}{4}\rho_M$ for $x \in (0, 1/2]$ and $\rho^0(x) = \frac{3}{4}\rho_M + 10^{-4}$ for $x \in (1/2, 1]$, $u^0(x) = u^{eq}(\rho^0(x))$. We use

$$\delta(\rho) = \delta_0 \frac{u^{eq}(\rho)}{u_f}$$

with $\delta_0 = 10^{-4}$ [h]. In Fig. 1, from the small initial perturbation we see linear instabilities that rapidly grow in time. Using the same initial data, in Fig. 2 the growth of nonlinear instabilities is observed using $\alpha = 1$ and $\ell = 10^{-7}$ [km]. In a last test (Fig. 3), we show the influence of the relaxation parameter in the case $\alpha = 1$ for the piecewise constant initial condition $\rho^0(x) = \frac{1}{2}\rho_M$ for $x \in (0, 1/2]$ and $\rho^0(x) = \frac{3}{4}\rho_M$ for $x \in (1/2, 1]$, $u^0(x) = u^{eq}(\rho^0(x))$. We here use $\rho_0 = \rho_M$ and the relaxation parameter

$$\ell(\rho) = \ell_0 \frac{u^{eq}(\rho)}{u_f}$$

Solutions are shown at time $t = 0.1$. From subplot (a) to (d), four different values of ℓ_0 are used, namely 10^{-7}, 10^{-6}, 10^{-5} and 10^{-4} km respectively. For $\ell_0 = 10^{-7}$, we find a stable discrete solution very close to the equilibrium solution computed by the first-order LWR model. For $\ell_0 = 10^{-6}, 10^{-5}$, one can see small oscillations that develop in the dense traffic region. Finally, for the larger value $\ell_0 = 10^{-4}$, the instabilities become important in the dense traffic region. From this experiment, one can see that the relaxation parameter can be adjusted in order to give the expected rate of instabilities.

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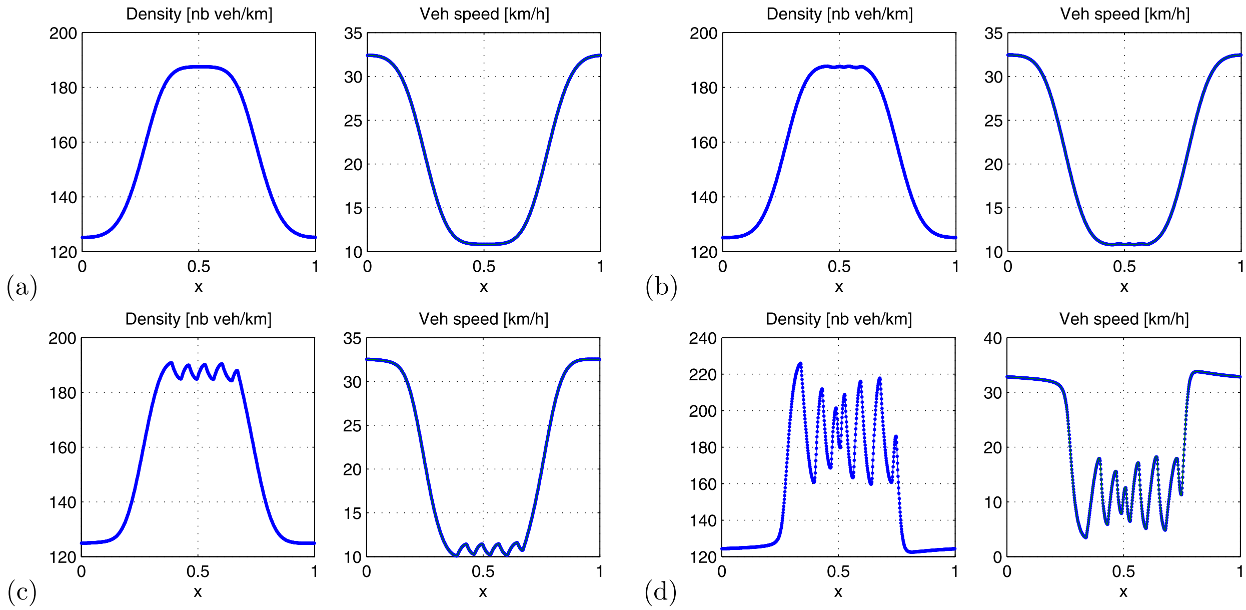

Fig. 3. Effect of the relaxation parameter, case $\alpha = 1$. Density and mean speed profiles at time $t = 1$, (a) for $\ell = 10^{-7}$, (b) for $\ell = 10^{-6}$, (c) for $\ell = 10^{-5}$ and (d) for $\ell = 10^{-4}$. The relaxation parameter can be adjusted in order to give the expected rate of instabilities.

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